

# Rotational glass transitions and jamming without quenched disorder in a large dimensional limit

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We study glass transitions and jamming of supercooled vectorial spin systems without quenched disorder in a large dimensional limit. The theory provides a unified mean-field theoretical framework for glass transitions of rotational degree of freedoms such as color angles in the continuous coloring of graphs, vector spins of geometrically frustrated magnets and directors of Janus particles and ellipsoids. The rotational glass transitions accompany various types of replica symmetry breaking. In the case of repulsive hardcore interactions in the spin space, the criticality of the jamming or SAT/UNSAT transition becomes the same as that of hardspheres.

*Introduction* – Various amorphous solid states exist in natural and industrial materials [1] providing a plenty of intriguing questions [2]. Oftenly the constituent elements such as molecules, colloids and granular particles have not only translational but also orientational degree of freedoms. Sometimes the rotational degree of freedoms alone exhibit glassiness. This happens for instance in the so called plastic crystals [3], Janus particles (Fig. 1c)[4], vectorial spins in frustrated magnets (Fig. 1b)[5, 6]. Their phenomenological aspects resemble to some extent those of the so called spinglasses for which useful theoretical insights can be obtained by studying the Edwards Anderson model [7]. However, in contrast to the spinglasses, they do not have build in quenched disorder. Possibilities of disorder-free spinglass transitions have been a matter of long debate.

On the other hand recently there has been a substantial theoretical progress of structural glasses without quenched disorder based on the replicated liquid theory which combines the density functional theory for liquids and the replica technique [8]. Most notably exact mean-field theory in large dimensional limit has been obtained for the hardspheres [9–15]. There it is understood that the glasses are born out of supercooled liquids from which they naturally inherit the disorder. Then one would naturally wonder that spinglasses without quenched disorder may possibly emerge similarly from supercooled spin-liquids or paramagnets in the absence of quenched disorder. In the present letter we substantiate this idea by developing and analyzing an exact mean-field theory of the glass transitions and jamming of the rotational degree of freedoms in large dimensional limits following the strategy used for the hardspheres. We first study vectorial spins put on lattices and later we consider those carried by particles like the Janus particles.

Analogous problems can be found in other contexts like the continuous coloring of graphs or periodic scheduling [16] (Fig. 1 a)). The problem is to put continuous colors parametrized by “color angle”  $0 < \theta < 2\pi$  on the vertexes of a given graph such that angles on adjacent vertexes are sufficiently separated from each other. This is exactly a continuous version of the usual coloring problem where one is allowed to use only discrete colors like red, green

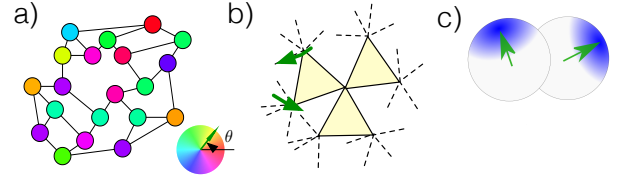


FIG. 1. Glassy systems carrying ‘spins’ representing rotational degree of freedoms. a) **Continuous coloring of a graph**: The color angle  $0 < \theta < 2\pi$ , as in the standard HSV color map, can be represented by a XY spin, i.e. a vector with  $M = 2$  component (green arrow). The example shown here is a solution to the requirement that color angle on adjacent vertexes must be greater than or equal to  $2\pi/3$ . b) **Geometrically frustrated magnets**: vectorial spins (green arrows) with antiferromagnetic couplings on adjacent vertexes on corner sharing triangles (e.g. kagome lattice), tetrahedras (e.g. pyrochlore lattice). The ground states are highly degenerate due to the loose connectivity of the lattices. c) **Janus particles**: hardspheres with attractive (white) and repulsive (blue) surfaces whose orientations are represented by the directors (green arrows).

and blue [17]. Upon increase of the coordination number  $c$  of the graph, the solution space exhibit clustering transition (glass transition) and then SAT/UNSAT transition (jamming) above which one cannot find a solution which satisfies the constraints. This is one of the constrained satisfaction problems of discrete variables [18]. It is then natural to expect that similar transitions also take place with continuous colors. Given the continuous variables, an interesting question is the universality class of the SAT/UNSAT transition. We will show that a class of  $p$ -spin models exhibit the same universality as that of hardspheres [11] extending the result on the perceptron problem studied by Franz and Parisi [19] which can be regarded as a special case  $p = 1$ .

*Vectorial spin model* – We consider vectorial spins with  $M$  components  $\mathbf{S}_i = (S_i^1, S_i^2, \dots, S_i^M)$  ( $i = 1, 2, \dots, N$ ), normalized such that  $|\mathbf{S}_i|^2 = \sum_{\mu=1}^M (S_i^\mu)^2 = M$ . We consider spins put on the vertexes of lattices (graphs) which are locally tree-like as shown in Fig. 5. The interaction between the spins is described by the

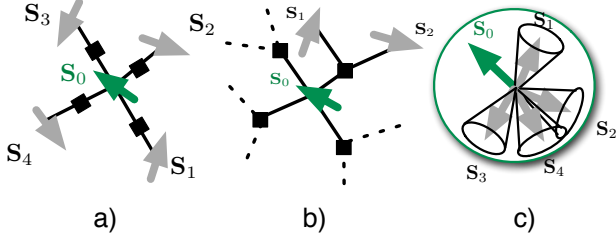


FIG. 2. A schematic picture of the model. Panel a):  $p = 2$ -body interaction on a graph with connectivity  $c = 4$ . Panel b):  $p = 3$ -body model with connectivity  $c = 4$ . The filled square represents the interactions. Panel c): the spin space of a spin  $\mathbf{S}_0$  with  $M = 3$  components (Heisenberg model). In the case of the hardcore potential Eq. (22), the spin  $\mathbf{S}_0$  is excluded from the cones around each of the neighboring spins.

generalized  $p$ -body interaction,

$$H = \sum_{\langle i_1, i_2, \dots, i_p \rangle} V(r(\mathbf{S}_{i_1}, \dots, \mathbf{S}_{i_p})) \quad (1)$$

where the summation is taken over sets of the interacting  $p$ -spins and  $V(r(\mathbf{S}_{i_1}, \dots, \mathbf{S}_{i_p}))$  is a generic interacting potential involving  $p$ -spins with

$$r(\mathbf{S}_{i_1}, \dots, \mathbf{S}_{i_p}) = \delta - \frac{1}{\sqrt{M}} \sum_{\mu=1}^M S_{i_1}^\mu \dots S_{i_p}^\mu \quad (2)$$

*Replicated spin liquid* – To focus on the possibility of glassy phases, we consider a system of replicas  $a = 1, 2, \dots, n$  with the Hamiltonian,  $H_n = \sum_{a=1}^n \sum_{\langle i_1, \dots, i_p \rangle} V(r(\mathbf{S}_{i_1}^a, \dots, \mathbf{S}_{i_p}^a))$ . There are no interactions between different replicas. The free-energy of the system can be written as,

$$-\beta F = \partial_n Z_n|_{n=0}, \quad Z_n = \int \mathcal{D}[\rho(\bar{\mathbf{S}})] e^{-\beta \mathcal{F}_n[\rho(\bar{\mathbf{S}})]} \quad (3)$$

where  $\beta = 1/k_B T$  is the inverse of the temperature  $T$  (we choose  $k_B = 1$ ) and  $\rho(\bar{\mathbf{S}}) \equiv N^{-1} \sum_{i=1}^N \prod_{a=1}^n \delta(\mathbf{S}^a - \bar{\mathbf{S}}_i^a)$  where we introduced replicated spin density with  $\bar{\mathbf{S}} \equiv (\bar{\mathbf{S}}^1, \bar{\mathbf{S}}^2, \dots, \bar{\mathbf{S}}^n)$ . Here  $\delta(\mathbf{S})$  is the delta function in the spin-space defined such that the integration over the spin space which is the surface of the  $M$ -dimensional sphere with diameter  $\sqrt{M}$  satisfies  $\int_S d\mathbf{S} \delta(\mathbf{S}) = 1$ . The integral  $\int \mathcal{D}[\rho(\bar{\mathbf{S}})] \dots$  is a functional integration over the spin density.

Given the locally tree-like lattice, the free-energy functional  $\mathcal{F}[\rho(\bar{\mathbf{S}})]$  introduced above can be expressed in the thermodynamic limit  $N \rightarrow \infty$  as

$$-\beta \frac{\mathcal{F}_n[\rho(\bar{\mathbf{S}})]}{N} = - \int_S d\bar{\mathbf{S}} \rho(\bar{\mathbf{S}}) \ln \rho(\bar{\mathbf{S}}) + \frac{c}{p} \int_S d\bar{\mathbf{S}}_1 \dots d\bar{\mathbf{S}}_p \rho(\bar{\mathbf{S}}_1) \dots \rho(\bar{\mathbf{S}}_p) f(\bar{\mathbf{S}}_1, \dots, \bar{\mathbf{S}}_p), \quad (4)$$

where  $f$  is the replicated Mayer function,

$$f(\bar{\mathbf{S}}_1, \bar{\mathbf{S}}_2, \dots, \bar{\mathbf{S}}_p) = \prod_{a=1}^n e^{-\beta V(r(\mathbf{S}_1^a, \dots, \mathbf{S}_p^a))} - 1. \quad (5)$$

We look for glassy states which emerge from the super-cooled spin liquid: spin configurations are just as disordered as the spin liquid state (paramagnetic state) such that the rotational invariance in the spin space remains. To this end we introduce the glass order parameter matrix  $\hat{Q}$  defined as the mutual overlap of spin patterns between replicas,

$$Q_{ab} = \lim_{N \rightarrow \infty} \frac{1}{NM} \sum_{i=1}^N \sum_{\mu=1}^M \langle (S^a)_i^\mu (S^b)_i^\mu \rangle \quad (6)$$

which is invariant under global rotations of spins in all replicas. Here  $\langle \dots \rangle$  represents the thermal average. Note also that  $Q_{aa} = 1$  due to the spin normalization.

It turns out that in the limit  $M \rightarrow \infty$  with fixed  $\alpha = c/M$  the free-energy of the system in the glassy phase can be expressed exactly as (see S. M. for the details),

$$-\beta \frac{F}{NM} = -\beta f[\hat{Q}^*] \quad (7)$$

Here  $\hat{Q}^*$  is the saddle point of the free-energy functional  $f[\hat{Q}]$  expressed exactly in terms of the glass order parameter matrix  $\hat{Q}$ ,

$$\begin{aligned} -\beta f[\hat{Q}] &= \partial_n s_n[\hat{Q}] \Big|_{n=0} \quad s_n[\hat{Q}] \equiv \frac{1}{2} \ln \det \hat{Q} - \frac{\alpha}{p} \mathcal{F}[\hat{Q}] \\ -\mathcal{F}[\hat{Q}] &\equiv e^{\frac{1}{2} \sum_{a,b=1}^n (Q_{ab})^p \partial_{h_a} \partial_{h_b}} \prod_{a=1}^n e^{-\beta V(\delta + h_a)} \Big|_{\{h_a=0\}} \end{aligned} \quad (8)$$

where we dropped off irrelevant constants. The saddle point  $\hat{Q}^*$  is determined by solving,  $\frac{\delta s[\hat{Q}]}{\delta Q_{ab}} \Big|_{\hat{Q}=\hat{Q}^*} = 0$ .

*A digression with quenched disorder* – Let us consider a system with quenched disorder replacing Eq. (2) by,

$$r(\mathbf{S}_{i_1}, \dots, \mathbf{S}_{i_p}, \xi_{i_1, i_2, \dots, i_p}^\mu) = \delta - \frac{1}{\sqrt{M}} \sum_{\mu=1}^M \xi_{i_1, i_2, \dots, i_p}^\mu S_{i_1}^\mu \dots S_{i_p}^\mu \quad (9)$$

Here  $\xi_{i_1, i_2, \dots, i_p}^\mu (= \xi_{i_2, i_1, \dots, i_p}^\mu = \dots)$  are random variables with Gaussian distribution with zero mean and unit variance. Remarkably the free-energy averaged over the quenched disorder becomes exactly the same as Eq. (66).

*Parisi's ansatz* – We assume the Parisi's ansatz for the order parameter matrix  $Q_{ab}$  at the saddle point parametrized by a function  $q(x)$  with  $0 < x < 1$  with the Edwards-Anderson (EA) order parameter, which we denote simply as  $q$ , located at the end  $x = 1$ , i. e.  $q = q(1)$  [20]. The free-energy functional is obtained as,

$$\begin{aligned} -\beta f[\hat{Q}] &= \frac{1}{2} \left[ \frac{q(0)}{G(0)} + \frac{1}{2} \ln G(1) + \int_0^1 \frac{dx}{G(x)} \right] \\ &+ \frac{\alpha}{p} \int \mathcal{D}z_0 (-f(0, \delta - \sqrt{q^p(0)} z_0)) \end{aligned} \quad (10)$$

with  $G(x) \equiv 1 - \int_x^1 dy q(y) - x q(x)$  and we introduced a shorthand notation  $\int \mathcal{D}z \dots = \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \dots$ .

The function  $f(x, h)$  obeys a partial differential equation,  $\partial_x f(x, h) = -\frac{\lambda(x)}{2} \left[ \partial_h^2 f(x, h) - x (\partial_h f(x, h))^2 \right]$  with  $\lambda(x) \equiv q^p(x)$ . The boundary condition is given by,  $f(1, h) = -\ln \int \mathcal{D}z e^{-\beta V(h - \sqrt{1-q^p(1)}z)}$ .

*Linear potential: recovery of  $p$ -spin spherical spinglass model* – So far we have not specified the potential  $V(x)$ . Before proceeding further let us consider here the simplest case: the linear potential, i.e.

$$V(x) = Jx, \quad (11)$$

The interaction part of the free-energy Eq. (66) becomes,  $-\mathcal{F}[\hat{Q}] = e^{\frac{(\beta J)^2}{2} \sum_{a,b=1}^n Q_{ab}^p}$ . Then it is easy to see that the free-energy Eq. (66) is exactly the same as that of the  $p$ -spin spherical spinglass model [21] (the factor  $\mu = p(\beta J)^2/2$  in equation (3.17) of [21] corresponds to  $2\alpha$ ). It is known that  $p = 2$  spherical model exhibits a phase transition but without replica symmetry breaking [22]. On the other hand  $p > 2$  system exhibit 1 step replica symmetry breaking [21]. The latter models show the essence of the glass phenomenology such as the dynamical and static glass transitions so that they are regarded as prototypical theoretical model to capture the physics of structural glasses [23]. Actually the quenched disorder is unnecessary as we show here.

*Non-linear potential* – Richer physics may emerge if the potential becomes non-linear. For instance a simple quadratic potential,

$$V(x) = \epsilon x^2 \quad \epsilon > 0, \quad (12)$$

allows RSB even with  $p = 2$  because it generates higher order terms like  $Q_{ab}^4$  in the free-energy. Then the situation is similar to the  $2 + p$  spherical model [24] which exhibits various types of RSB.

We will focus on a more strongly non-linear potential,

$$V(x) = \epsilon x^2 \theta(-x) \quad \epsilon > 0, \quad (13)$$

which becomes a hardcore potential in the limit  $\epsilon \rightarrow \infty$ . With  $p = 2$  body interaction it can be used for the continuous coloring problem shown in Fig. 1 a): spins representing the color angles on adjacent vertexes are forced to be separated in angle *larger than*  $\cos^{-1}(\delta/\sqrt{M})$  for the hardcore potential (See Fig. 5 c)). In the case of  $p = 1$ , and in the presence of quenched disorder  $\xi^\mu$ 's Eq. (9), the problem becomes the perceptron problem [25] [19]. We now explicitly study the case of  $p = 2$  [26].

*Liquid (paramagnetic) phase and its stability* – We naturally expect that the normal liquid (paramagnetic) phase exists where the replica symmetric (RS) ansatz with  $q(x) = q = 0$  ( $0 < x < 1$ ) is the relevant solution. The stability of the solution can be studied by analyzing the stability matrix (Hessian matrix). In the case of  $p = 2$  body we find the eigenvalues of the Hessian matrix associated with the  $q = 0$  RS solution vanishes as  $\alpha \rightarrow \alpha_c^-$  at the critical point  $\alpha_c(\delta, T) = (\int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} (e^{-\beta V(\delta-z)})' / \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} e^{-\beta V(\delta-z)})^{-2}$ .

The situation is the same as the d'Almeida-Thouless (AT) instability [27] of the Sherrington-Kirkpatrick model. This implies a continuous phase transition at  $\alpha_c$  to a different phase in contrast to the random first order (RFOT) scenario [23, 28] which is established for hardspheres in  $d \rightarrow \infty$  limit [9]. On the other hand for  $p > 2$ , eigenvalues are always positive so that different phases, if any, would emerge discontinuously.

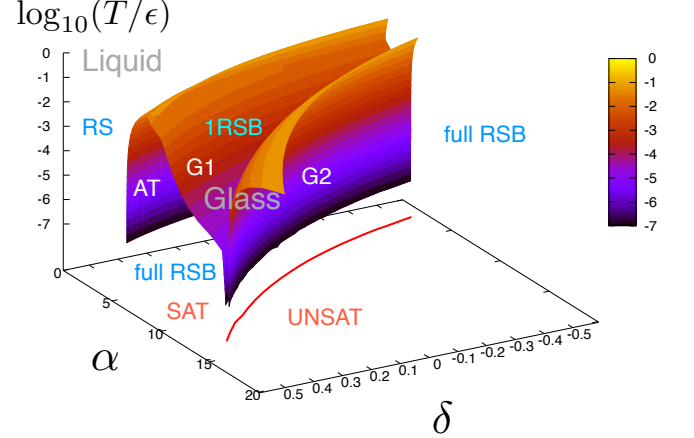


FIG. 3. Phase diagram of the soft/hardcore model ( $p = 2$ ). On the plane AT separating the liquid (RS) and glass (RSB) the d'Almeida-Thouless (AT) instability occurs. The 1RSB solution becomes unstable below the two planes G1 and G2 on which the Gardner transition occurs. The G1 plane separates from the AT plane at finite temperatures. The red line on the bottom represents the jamming line  $\alpha = \alpha_J(\delta)$  at  $T = 0$ .

*Glass phase and jamming* – Let us now study the glass phase for the case of the hard/softcore potential Eq. (22) focusing on the  $p = 2$  case. By analyzing the solutions of the saddle point equations we obtain the phase diagram as shown in Fig. 3. The boundary between the liquid and glass phase is given by the plane  $\alpha = \alpha_c(\delta, T)$  where the AT instability occurs. There is a region where we find stable 1 step replica symmetry breaking (1RSB) solution. The 1RSB solution is parametrized by the EA order parameter  $q$  and  $m$  such that  $q(x) = q$  for  $m < x < 1$  and  $q(x) = 0$  for  $0 < x < m$ . The EA order parameter  $q$  develops continuously passing the AT plane. We find the Gardner transition [29] on the two planes denoted as G1 and G2 in Fig. 3 where the 1RSB solution becomes unstable. This suggests further breaking of the residual replica symmetry. Indeed we obtain the full RSB solution as shown in Fig. 4.

The liquid phase at  $T = 0$  can be regarded as an easy SAT region where the space of the solutions to satisfy the hard constraints ( $\epsilon \rightarrow \infty$ ), i. e. the manifold of the ground states are continuously connected. The glass phase at  $T = 0$  with  $q(1) < 1$  can be regarded as hard SAT phase where the manifold of the ground states splits into clusters [17]. On the  $T = 0$  plane we find “jammed” or UNSAT region where the EA order parameter  $q = q(1)$

is saturated to 1 meaning absence of thermal fluctuations. We find the jamming line, i. e. the border line  $\alpha = \alpha_j(\delta)$  between the SAT/UNSAT regions by solving the saddle point equations directly in the limit  $q(1) \rightarrow 1$ . Interestingly we find that the two planes of the Gardner transition (G1 and G2) merge onto the jamming line in  $T \rightarrow 0$  limit (see Fig. 3).

We analyzed the criticality of jamming  $q(1) \rightarrow 1$  of the hardcore model ( $\epsilon \rightarrow \infty$ ), i. e.  $\alpha \rightarrow \alpha_j^-(\delta)$  at  $T = 0$ . As shown in panel b) of Fig. 4, we find power law behavior  $1 - q(x) \propto x^{-\kappa}$ . We checked that analytical scaling argument for the case of hardspheres [11] and perceptron [19] that corresponds to  $p = 1$  of our model, can be extended to the present model predicting that the criticality is the same as that of the hardspheres for *any*  $p$  (See S. M. for the details).

Mean-field theory for frustrated magnets on corner sharing triangles or tetrahedras can be developed by extending the present approach on the networks of triangles (or tetrahedras) (see Fig. 1 b)). In the simplest case of nearest neighbor interactions the strong geometrical frustration leads to extensive degeneracy of the ground states [30]. We can naturally ask possibility of glass transitions due to clustering of the ground state manifold much as in the case of the coloring problems [17]. Detailed discussion goes beyond the scope of the present paper so that we report it elsewhere.

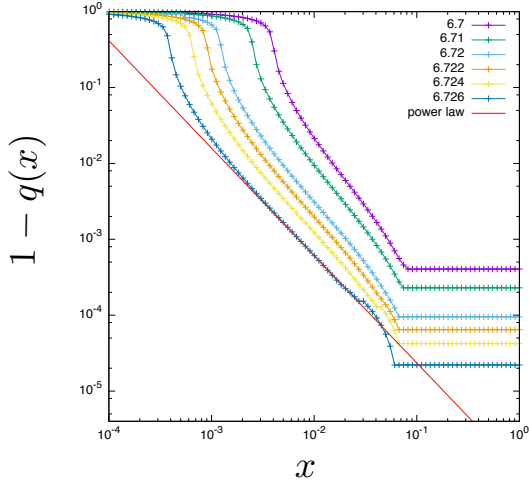


FIG. 4. The  $q(x)$  function of the hardcore model with  $p = 2$ ,  $\delta = 0$  for which  $\alpha_c = 1.5708..$  and  $\alpha_j = 6.732..$  The straight line represents the power law fit  $ax^{-\kappa}$  with  $\kappa = 1.4157$ , the same exponent as that for the hardspheres[11].

*Anisotropic particles* – Finally let us consider an assembly of *anisotropic* particles in  $d$ -dimensional space interacting with each other through a two-body potential,

$$H = \sum_{i < j} V(\mathbf{r}_{ij}, \mathbf{S}_i, \mathbf{S}_j), \quad (14)$$

where  $\mathbf{r}_i$  and  $\mathbf{S}_i$  ( $i = 1, 2, \dots, N$ ) are  $d$ -dimensional vectors representing the position and *shape* of the particles.

The latter is, for instance, the director of Janus particles (See Fig. 1 c)). We assume the system is rotationally invariant and the potential is parametrized as,

$$V(\mathbf{r}_{12}, \mathbf{S}_1, \mathbf{S}_2) = V(r_{12}, \hat{\mathbf{r}}_{12} \cdot \mathbf{S}_1, \hat{\mathbf{r}}_{12} \cdot \mathbf{S}_2, \mathbf{S}_1 \cdot \mathbf{S}_2) \quad (15)$$

where  $\hat{\mathbf{r}} = \mathbf{r}/r$  and  $r = |\mathbf{r}|$ .

We can extend our analysis of the spin system on the lattice to obtain the exact free-energy functional of the replicated liquid of the anisotropic particles. First we decompose the coordinates of the particles as  $\mathbf{r}_i^a = \mathbf{R}_i + (D/\sqrt{d})\boldsymbol{\eta}_i^a$  where  $D$  represents the size of the particle and  $\mathbf{R}_i$  represents the center of mass coordinate of the 'molecule' of replicas. We denote the glass order parameter for the translational degree of freedoms [10] as  $\alpha_{ab} = \langle \boldsymbol{\eta}_i^a \cdot \boldsymbol{\eta}_i^b \rangle$ . Because of the identity  $\sum_a \boldsymbol{\eta}_i^a = 0$ , the matrix  $\hat{\alpha}$  can be fully specified by a submatrix  $\hat{\alpha}^{mm}$  which is defined by subtracting the  $m$ -th row and column of the former. The orientational glass order parameter is  $Q_{ab} = (1/d)\langle \mathbf{S}_i^a \cdot \mathbf{S}_i^b \rangle$  as before Eq. (40). We also introduce  $\beta_{ab} = \beta_{ba} = (1/\sqrt{d})\langle \boldsymbol{\eta}_i^a \cdot \mathbf{S}_i^b \rangle$  which represents the correlation between the rotational/translational fluctuations. For convenience we also introduce a matrix  $\hat{Q}_{\text{tot}}$  which is a supermatrix consisting of  $\hat{\alpha}$ ,  $\hat{\beta}$  and  $\hat{Q}$ . The free-energy of the replicated liquid in the  $d \rightarrow \infty$  limit is obtained as,

$$-\beta \frac{F[\hat{Q}_{\text{tot}}]}{N} = 1 - \ln \rho + d \ln m + \frac{(2m-1)d}{2} \ln \left( \frac{2\pi e}{d} \right) + \frac{d}{2} \ln \det \hat{Q}_{\text{tot}}^{m,m} - \frac{d}{2} \hat{\varphi} \mathcal{F}[\hat{Q}_{\text{tot}}] \quad (16)$$

where  $\rho$  is the number density,  $\hat{\varphi} = 2^d \varphi/d$  is the reduced volume fraction with  $\varphi$  being the volume fraction and

$$-\mathcal{F}[\hat{Q}_{\text{tot}}] = \int_{-\infty}^{\infty} d\xi e^{\xi} e^{\frac{1}{2} \sum_{a,b=1}^m \mathcal{D}_{ab} f_m(\bar{\xi}, \bar{x}, \bar{x}', \bar{h})} \Big|_{\substack{\{\bar{\xi}=\xi\} \\ \bar{x}=\bar{x}'=\bar{h}=0}} \\ \mathcal{D}_{ab} \equiv -\Delta_{ab} \partial_{\xi_a} \partial_{\xi_b} + \beta_{ab} (\partial_{\xi_a} + \partial_{\xi_b}) (\partial_{x_b} - \partial_{x'_b}) + Q_{ab} (\partial_{x_a} \partial_{x_b} + \partial_{x'_a} \partial_{x'_b}) + (Q_{ab})^2 \partial_{h_a} \partial_{h_b} \\ f_m(\bar{\xi}, \bar{x}, \bar{x}', \bar{h}) \equiv \prod_{a=1}^m e^{-\beta V(D(1+\frac{\xi_a}{d}), x_a, x'_a, h_a)} - 1 \quad (17)$$

with  $\Delta_{ab} = \alpha_{aa} + \alpha_{bb} - 2\alpha_{ab}$ . (See S. M. for the details.) We see that the free-energy of spheres [10] is recovered if the spins are absent.

As a prototypical system let us consider the Janus particles (See Fig. 1 c)) whose interaction potential can be modeled by the Kern-Frenkel potential [31],

$$e^{-\beta V(\mathbf{r}, \mathbf{S}_1, \mathbf{S}_2)} = e^{-\beta U(r)} \theta(r-D) \theta(\hat{\mathbf{r}} \cdot \mathbf{S}_1 - \delta) \theta(-\hat{\mathbf{r}} \cdot \mathbf{S}_2 - \delta) \quad (18)$$

Here  $\theta(r-D)$  and  $U(r)$  represents the hardcore repulsion and the short-range attraction. In this case the translational and orientational degree of freedoms can be decoupled allowing the spins to exhibit glass transition and jamming separately from the translational ones: the spins behave essentially as in the  $p = 1$  spin model Eq. (66) with the hardcore interaction Eq. (22).

More generally, as in ellipsoids, the translational  $\xi$ , and orientational variables  $x, x'$  and  $h$  can be coupled [32, 33]. It will be very useful to work out the details of translational/orientational glass transitions of various cases by the present approach.

To conclude we studied glass transition and jamming of a class of supercooled vectorial spin systems developing and analyzing a mean-field theory which becomes exact in the large dimensional limit. We believe the present work provides a useful starting point for further developments for various problems including orientational glass transition and jamming, frustrated magnets and continuous constrained satisfaction problems.

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# Supplementary material

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## I. VECTORIAL SPIN MODEL

We consider a family of vectorial spin models on lattices (graph) with  $N$  vertexes onto which spins  $\mathbf{S}_i$  ( $i = 1, 2, \dots, N$ ) are put. The spins are  $M$ -component vectors  $\mathbf{S}_i = (S_i^1, S_i^2, \dots, S_i^M)$  normalized such that  $|\mathbf{S}_i|^2 = \sum_{\mu=1}^M (S_i^\mu)^2 = M$ . The Hamiltonian is given by,

$$H = \sum_{\langle i_1, i_2, \dots, i_p \rangle} V(r(\mathbf{S}_{i_1}, \dots, \mathbf{S}_{i_p})) \quad (19)$$

where the summation is took over sets of the interacting  $p$ -spins and  $V(r(\mathbf{S}_{i_1}, \dots, \mathbf{S}_{i_p}))$  is a generic interacting potential involving  $p$ -spins with

$$r(\mathbf{S}_{i_1}, \dots, \mathbf{S}_{i_p}) = \delta - \frac{1}{\sqrt{M}} \sum_{\mu=1}^M S_{i_1}^\mu \dots S_{i_p}^\mu \quad (20)$$

which we call as 'gap' in the following. The free-energy  $F$  and the partition function  $Z$  of the system is given by,

$$e^{-\beta F} = Z = \int_S \prod_{i=1}^N d\mathbf{S}_i \prod_{\langle i_1, i_2, \dots, i_p \rangle} e^{-\beta V(r(\mathbf{S}_{i_1}, \dots, \mathbf{S}_{i_p}))} \quad (21)$$

where  $\beta$  is the inverse temperature and  $\int_S d\mathbf{S}$  stands for an integration over the spin-space under the constraint  $|\mathbf{S}|^2 = M$ , which is the surface of a  $M$ -dimensional sphere with radius  $\sqrt{M}$ .

In the present paper we limit ourselves with mean-field models in the following two respects:

- We consider lattices (graphs) which look locally like a tree such that effect of loops can be neglected. This holds true for instance in hyper-cubic lattices in large spatial dimension  $d \rightarrow \infty$  or Bethe like lattices. More precisely we consider that each spin participates into  $c$  sets of  $p(\geq 2)$ -body interactions. In Fig. 5, the interaction is represented by the interaction node (filled squares). To summarize, there are  $N$  vertexes (variable nodes) labeled as  $i = 1, 2, \dots, N$  and each of them carries a spin  $\mathbf{S}_i$  with  $M$  components. Each variable node has  $c$  arms which are connected to different interaction nodes. There are  $Nc/p$  interaction nodes and each of them has  $p$  arms which are connected to different variable nodes.
- We consider vector spins with large number of components  $M \rightarrow \infty$  and lattices with large number of connectivity  $c \rightarrow \infty$  with fixed ratio  $\alpha = c/M$ . In this limit we will find that the theoretical analysis becomes much easier.

We will study in particular the case of soft/hardcore repulsive contact potential,

$$V(x) = \lim_{\epsilon \rightarrow \infty} \epsilon x^2 \theta(-x). \quad (22)$$

The hardcore potential is obtained in  $\epsilon \rightarrow \infty$  limit. This amount to bring excluded volume effect in the spin space similarly to the interaction between the hardspheres (See Fig. 5 c)). Under this potential, the system becomes more constrained as we decrease the parameter  $\delta$  much as an assembly of hardspheres becomes more constrained as the

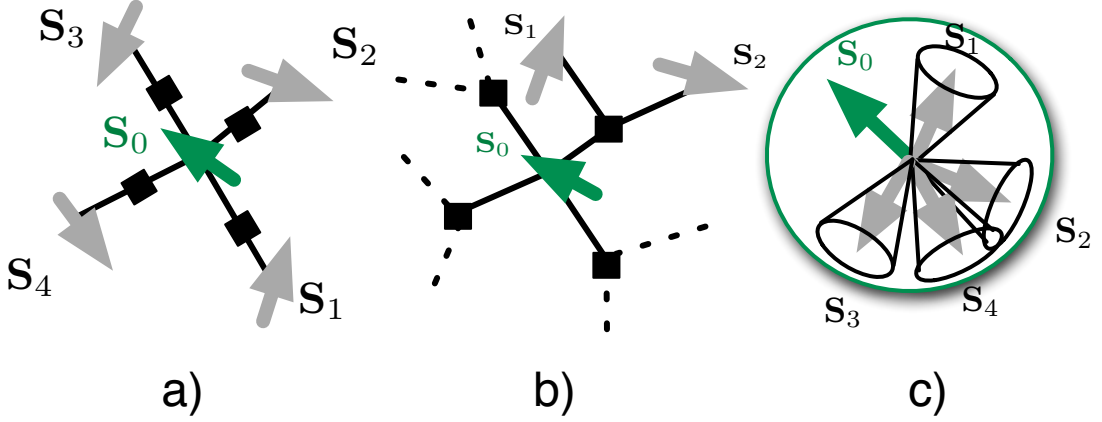


FIG. 5. A schematic figure of the model. Panel a) and b) is for the case of  $p = 2$ -body interaction on a graph with connectivity  $c = 4$ . Vectorial spins with  $M$  components, in this example  $M = 3$  (Heisenberg spins), are put on the vertexes of a lattice or a graph as shown in the left panel a). Panel b) is for the  $p = 3$ -body model with connectivity  $c = 4$ . The filled square represents the interaction nodes each of which connects a set of three spins on the vertexes (variable nodes) interacting with each other. For the hardcore potential Eq. (22) the spin  $\mathbf{S}_0$  in panel c) is excluded from the cones around each of the neighboring spins  $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ . (Note that, for instance,  $\mathbf{S}_2$  and  $\mathbf{S}_4$  can overlap if they are not directly connected by a link). The size of the cones grows with decreasing the parameter  $\delta$ . Thus the excluding volume effect becomes larger by decreasing  $\delta$  or increasing the connectivity  $c$ .

diameter of the spheres increase so that the volume fraction increases. This motivates us to introduce 'pressure' as an analogue of that in particulate systems,

$$\Pi = -\frac{p}{cN} \frac{\partial \beta F}{\partial \delta}. \quad (23)$$

The normalization factor  $cN/p$  is simply the number of interaction links in the system. Then it is also useful to introduce the distribution function of the gap Eq. (20),

$$g(r) \equiv \left\langle \delta(r - r(\mathbf{S}_{i_1}, \dots, \mathbf{S}_{i_p})) \right\rangle \quad (24)$$

$$= \frac{p}{cN} \frac{\delta(-\beta F)}{\delta \ln e^{-\beta V(r)}} \quad (25)$$

In the 1st equation  $\langle \dots \rangle$  is the thermal average. In the 2nd equation  $\delta/\delta \ln e^{-\beta V(r)}$  is a functional derivative. Apparently the distribution function of the gap  $g(r)$  is analogous to the radial distribution function in the particulate systems. The pressure Eq. (23) can be rewritten using  $\partial(-\beta F)/\partial \delta = \int dr \frac{\delta(-\beta F)}{\delta \ln e^{-\beta V(r)}} (\ln e^{-\beta V(r)})'$  and  $g(r)$  defined above as,

$$\Pi = \int dr g(r) (\ln e^{-\beta V(r)})' = \int dr g(r) (-\beta V'(r)). \quad (26)$$

This is the analogue of the virial equation for the pressure in the liquid theory [34].

Given  $N$  spins  $\mathbf{S}_i$  ( $i = 1, 2, \dots, N$ ) with  $M$  components, which are normalized such that  $|\mathbf{S}_i|^2 = M$ , the total number of the degrees of freedom is  $N(M-1)$ . Each spin is involved in  $c = \alpha M$  sets of  $p$ -body interactions (See Fig. 5). We say the gap associated with such an interaction is *closed* if  $r(\mathbf{S}_{i_1}, \dots, \mathbf{S}_{i_p}) < 0$ . The fraction of the interactions or contacts whose gaps are closed can be written as

$$f_{\text{closed}} = \lim_{\epsilon \rightarrow 0} \int_0^\epsilon dr g(r) \quad (27)$$

where  $g(r)$  is the distribution function of the gap defined in Eq. (25). This means there are  $N(c/p)f_{\text{closed}}$  constraints. Then isostaticity implies,

$$M-1 = \frac{c}{p} \lim_{\epsilon \rightarrow 0} \int_0^\epsilon dr g(r). \quad (28)$$



## II. REPLICATED SPIN LIQUID

### A. Density functional

#### 1. Spin liquid

Let us introduce 'spin' density defined as,

$$N\rho(\mathbf{S}) = \sum_{i=1}^N \delta(\mathbf{S} - \mathbf{S}_i) \quad (29)$$

where  $\delta(\mathbf{S})$  is the delta function in the spin-space which satisfies  $\int_S d\mathbf{S} \delta(\mathbf{S}) = 1$ . Let us also introduce the Mayer function,

$$f(\mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_p) = e^{-\beta V(\mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_p)} - 1 = -1 + \exp \left[ -\beta V \left( \delta - \frac{1}{\sqrt{M}} \sum_{\mu=1}^M \prod_{l=1}^p (S_l)^\mu \right) \right]. \quad (30)$$

A convenient strategy is to write the free-energy as,

$$e^{-\beta F} = \int \mathcal{D}[\rho(\mathbf{S})] e^{-\beta \mathcal{F}[\rho(\mathbf{S})]} \quad (31)$$

where we introduced a functional  $\mathcal{F}[\rho]$ ,

$$e^{-\beta \mathcal{F}[\rho(\mathbf{S})]} \equiv \int_S \prod_{i=1}^N d\mathbf{S}_i \prod_{\langle i_1, i_2, \dots, i_p \rangle} e^{-\beta V(S_{i_1}, S_{i_2}, \dots, S_{i_p})} \delta \left[ \rho(\mathbf{S}) - N^{-1} \sum_{i=1}^N \delta(\mathbf{S} - \mathbf{S}_i) \right] \quad (32)$$

with  $\int \mathcal{D}[\rho(\mathbf{S})]$  being a functional integration over  $\rho(\mathbf{S}) > 0$  and  $\delta[\dots]$  is a functional delta function.

To obtain the functional  $\mathcal{F}[\rho]$  one can follow the standard step of the liquid theory [34]: one defines first a free-energy  $F[\phi(\mathbf{S})]$  of the system with modified Hamiltonian  $H = \sum_{\langle i_1, i_2, \dots, i_p \rangle} V(\mathbf{S}_{i_1}, \mathbf{S}_{i_2}, \dots, \mathbf{S}_{i_p}) + \int_S d\mathbf{S} \rho(\mathbf{S}) \phi(\mathbf{S})$  then perform a Legendre transformation to obtain a free-energy as functional of the spin density  $\mathcal{F}[\rho(\mathbf{S})] = F[\phi(\mathbf{S})] - \int_S d\mathbf{S} \rho(\mathbf{S}) \phi(\mathbf{S})$ . As the result one finds,

$$-\beta \frac{\mathcal{F}[\rho(\mathbf{S})]}{N} = - \int_S d\mathbf{S} \rho(\mathbf{S}) \ln \rho(\mathbf{S}) + \frac{c}{p} \int_S d\mathbf{S}_1 d\mathbf{S}_2 \cdots d\mathbf{S}_p \rho(\mathbf{S}_1) \rho(\mathbf{S}_2) \cdots \rho(\mathbf{S}_p) f(\mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_p). \quad (33)$$

The free-energy  $F$  is obtained by minimizing the variational free-energy functional  $\mathcal{F}[\rho(\mathbf{S})]$ . The 1st term on the r.h.s of Eq. (33) represents the entropic (paramagnetic) part of the free-energy. The 2nd term is the 1st virial correction due to interactions. The reason for the absence of the higher order terms, all of which are represented as 1 particle irreducible (1PI) diagrams such as a triangle, a square, e.t.c. [34], is the tree-like geometry of the lattices that we consider.

#### 2. Replicated spin liquid

In principle all stable and metastable states of the system, including liquid (paramagnetic) state  $\rho_{\text{liq}}(\mathbf{S})$ , crystalline state  $\rho_{\text{crystal}}(\mathbf{S})$  and glassy states  $\rho_\alpha(\mathbf{S})$  ( $\alpha = 1, 2, \dots$ ), would be found as local minima of the free-energy functional Eq. (33). In the present paper we focus on the properties of glassy states which emerge from supercooled paramagnetic state.

A useful way to analyze the properties of glassy states is the replica approach. We consider replicated spin liquid of  $n$  replicas labeled as  $a = 1, 2, \dots, n$  obeying the Hamiltonian,

$$H_n = \sum_{a=1}^n \sum_{\langle i_1, i_2, \dots, i_p \rangle} V(\mathbf{S}_{i_1}^a, \mathbf{S}_{i_2}^a, \dots, \mathbf{S}_{i_p}^a) \quad (34)$$

For convenience we introduce a short hand notation

$$\bar{\mathbf{S}} = (\mathbf{S}^1, \mathbf{S}^2, \dots, \mathbf{S}^n) \quad (35)$$

where  $\mathbf{S}$ s themselves are  $M$  component spin vectors. Introducing replicated spin density

$$N\rho(\bar{\mathbf{S}}) = \sum_{i=1}^N \prod_{a=1}^n \delta(\mathbf{S}^a - \mathbf{S}_i^a) \quad (36)$$

which is normalized such that  $\int d\bar{\mathbf{S}} \rho(\bar{\mathbf{S}}) = 1$ , we find,

$$-\beta F = \partial_n Z_n|_{n=0} \quad Z_n = \int \mathcal{D}[\rho(\bar{\mathbf{S}})] e^{-\beta \mathcal{F}_n[\rho(\bar{\mathbf{S}})]} \quad (37)$$

with the variational replicated free-energy functional defined as,

$$-\beta \frac{\mathcal{F}_n[\rho(\bar{\mathbf{S}})]}{N} = - \int_S d\bar{\mathbf{S}} \rho(\bar{\mathbf{S}}) \ln \rho(\bar{\mathbf{S}}) + \frac{c}{p} \int_S d\bar{\mathbf{S}}_1 d\bar{\mathbf{S}}_2 \cdots d\bar{\mathbf{S}}_p \rho(\bar{\mathbf{S}}_1) \rho(\bar{\mathbf{S}}_2) \cdots \rho(\bar{\mathbf{S}}_p) f_n(\bar{\mathbf{S}}_1, \bar{\mathbf{S}}_2, \dots, \bar{\mathbf{S}}_p). \quad (38)$$

where  $d\bar{\mathbf{S}} = \prod_{a=1}^n d\mathbf{S}^a$  and we introduced a replicated Mayer function,

$$f_n(\bar{\mathbf{S}}_1, \bar{\mathbf{S}}_2, \dots, \bar{\mathbf{S}}_p) = -1 + e^{-\beta \sum_{a=1}^n V(\mathbf{S}_1^a, \mathbf{S}_2^a, \dots, \mathbf{S}_p^a)} = -1 + \prod_{a=1}^n \exp \left[ -\beta V \left( \delta - \frac{1}{\sqrt{M}} \sum_{\mu=1}^M \prod_{l=1}^p (S^a)_l^\mu \right) \right]. \quad (39)$$

## B. Glass order parameter functional

### 1. Glass order parameter

We look for glassy metastable states which keep the statistical rotational invariance of the liquid (paramagnetic) state. Then similarly to the Edwards-Anderson model for spin glasses [7], a natural candidate of the order parameter of the glassy states is the overlap matrix  $\hat{Q}$  defined as the mutual overlap of spin patterns between different replicas,

$$\hat{Q}_{ab} = Q_{ab} \quad Q_{ab} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \frac{1}{M} \sum_{\mu=1}^M (S^a)_i^\mu (S^b)_i^\mu \quad (40)$$

for  $a, b = 1, 2, \dots, n$ . Note that this is invariant under global rotation of the spins in the spin space. Note also that  $Q_{aa} = 1$  due to the normalization of the spins  $|\mathbf{S}_i^a|^2 = M$ .

### 2. Variational free-energy

Based on the above discussion we expect that  $\rho(\bar{\mathbf{S}})$  of the the glassy states, which keeps the statistical rotational invariance of the liquid, is parametrized solely by the overlap matrix  $\hat{Q}$ ,

$$\rho(\bar{\mathbf{S}}) = \rho(\hat{Q}). \quad (41)$$

Similarly we anticipate that the replicated Mayer function can be parametrized as,

$$f_n(\bar{\mathbf{S}}_1, \bar{\mathbf{S}}_2, \dots, \bar{\mathbf{S}}_p) = f_n(\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_p). \quad (42)$$

so that the variational free-energy functional Eq. (38) as a whole can be cast into the following rotationally invariant form,

$$\begin{aligned} -\beta \frac{\mathcal{F}_n[\rho(\hat{Q})]}{N} = & - \int d\hat{Q} J(\hat{Q}) \rho(\hat{Q}) \ln \rho(\hat{Q}) + \frac{c}{p} \int \prod_{l=1}^p \{d\hat{Q}_l J(\hat{Q}_l) \rho(\hat{Q}_l)\} f_n(\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_p) \\ & + \lambda \left( \int d\hat{Q} J(\hat{Q}) \rho(\hat{Q}) - 1 \right) \end{aligned} \quad (43)$$

where  $d\hat{Q} = \prod_{a < b} dQ_{ab}$ . Here  $J(\hat{Q})$  is the Jacobian (see below) and the parameter  $\lambda$  in the last term of Eq. (43) is a Lagrange multiplier to enforce the normalization of the spin density. Note that in the 2nd integral on the r. h. s. of Eq. (43) we assumed a simply factorized Jacobian  $\prod_{l=1}^p J(\hat{Q})$  disregarding possible cross-correlations of spins at different sites  $l = 1, 2, \dots, p$ . We comment on the validity of this assumption later.

The Jacobian  $J$  is defined as

$$J(\hat{Q}) \equiv \int d\bar{\mathbf{S}} \prod_{a \leq b} \delta \left( Q_{ab} - \frac{1}{M} \sum_{\mu=1}^M (S^a)^\mu (S^b)^\mu \right) \quad (44)$$

Here we have replaced the constrained integral  $\int_S d\bar{\mathbf{S}}$  by an unconstrained integral  $\int d\bar{\mathbf{S}} \equiv \prod_{\mu=1}^M \prod_{a=1}^n \int_{-\infty}^{\infty} d(S^a)^\mu$ . This is made possible by setting

$$Q_{aa} = 1 \quad (45)$$

for all  $a$  in Eq. (44) so that the normalization condition of the spins  $|\mathbf{S}^2| = M$  is enforced. Then one can evaluate the Jacobian to find (see Eq.(17) and (78) of [9]),

$$J(\hat{Q}) = C_{n+1,M} e^{\frac{1}{2}(M-(n+1)) \ln \det \hat{Q}} \quad (46)$$

with  $C_{n,M}$  being a numerical prefactor which behaves for  $M \gg 1$  as,

$$\ln C_{n,M} = \frac{M}{2}(n-1) \ln(2\pi e) - \frac{M}{2}(n-1) \ln M \quad M \gg 1. \quad (47)$$

Minimization of the variational free-energy Eq. (43) with respect to  $\rho(\hat{Q})$ ,

$$0 = \frac{\delta}{\delta \rho(\hat{Q})} \beta \frac{\mathcal{F}_n[\rho(\hat{Q})]}{N} \quad (48)$$

yields,

$$\ln \rho(\hat{Q}) = \lambda - 1 + c \int \prod_{l=1}^{p-1} \{d\hat{Q}_l J(\hat{Q}_l) \rho(\hat{Q}_l)\} f_n(\hat{Q}, \hat{Q}_1, \dots, \hat{Q}_{l-1}). \quad (49)$$

In addition, normalization of the spin density implies,

$$1 = \int d\hat{Q} J(\hat{Q}) \rho(\hat{Q}) = \int d\hat{Q} \exp \left( \ln J(\hat{Q}) + \ln \rho(\hat{Q}) \right) \quad (50)$$

with

$$\ln J(\hat{Q}) + \ln \rho(\hat{Q}) = \ln C_{n,M} + \frac{1}{2}(M-n) \ln \det \hat{Q} + \lambda - 1 + c \int \prod_{l=1}^{p-1} \{d\hat{Q}_l J(\hat{Q}_l) \rho(\hat{Q}_l)\} f_n(\hat{Q}, \hat{Q}_1, \dots, \hat{Q}_{l-1}) \quad (51)$$

where we used Eq. (46) and Eq. (49).

### 3. $M \rightarrow \infty$ limit

In the present paper we limit ourselves with the  $M \rightarrow \infty$  limit which greatly simplifies the analysis. The first advantage of the  $M \rightarrow \infty$  limit is that the integrals over  $\hat{Q}$  can be done by saddle point method. For example in Eq. (50) the saddle point value  $\hat{Q}^*$  is determined by,

$$0 = \frac{\delta}{\delta \hat{Q}} \left( \ln J(\hat{Q}) + \ln \rho(\hat{Q}) \right) \Big|_{\hat{Q}=\hat{Q}^*} \quad (52)$$

Importantly the integrals over  $\hat{Q}$  in the variational free-energy functional Eq. (43) can also be evaluated by the saddle point method in  $M \rightarrow \infty$  limit and the saddle point should be exactly the same as the one given by Eq. (52). This is because in the free-energy functional Eq. (43), only the factor  $J(\hat{Q})\rho(\hat{Q})$  is exponentially large in  $M$ . [9]

Here let us comment on the validity of our assumption used in Eq. (43) that fluctuations of spins at different sites  $l = 1, 2, \dots, p$  are uncorrelated which allowed us to assume a simply factorized Jacobian  $\prod_{l=1}^p J(\hat{Q})$  in the 2nd integral of Eq. (43). Actually more generally we should write the Jacobian as,

$$K(\{\hat{Q}_l\}, \{\hat{P}_{l'}\}) \equiv \int \prod_{l=1}^p d\mathbf{S}_l \prod_{a \leq b} \prod_l \delta \left( (Q_l)_{ab} - \frac{1}{M} \sum_{\mu=1}^M ((S_l)^a)^\mu ((S_l)^b)^\mu \right) \\ \times \prod_{l < l'} \delta \left( (P_{l'})_{ab} - \frac{1}{M} \sum_{\mu=1}^M ((S_l)^a)^\mu ((S_{l'})^b)^\mu \right) \quad (53)$$

where  $P_{l,l'}$  represents cross-correlation of the fluctuation of spins are different sites  $l$  and  $l'$ . Then similarly to Eq. (50) we may consider the normalization of the spin density,

$$1 = \int \prod_l d\hat{Q}_l \prod_{l < l'} d\hat{P}_{l'} \prod_{l=1}^p \rho(\{\hat{Q}_l\}) K(\{\hat{Q}_l\}, \{\hat{P}_{l'}\}). \quad (54)$$

which implies, similarly to Eq. (52),

$$0 = \frac{\delta}{\delta \hat{Q}_l} \left( \ln K(\{\hat{Q}_l\}, \{\hat{P}_{l'}\}) + \sum_{l=1}^p \ln \rho(\hat{Q}_l) \right) \Big|_{\hat{Q}=\hat{Q}^*} \quad l = 1, 2, \dots, p \\ 0 = \frac{\delta}{\delta \hat{P}_{l,l'}} \left( \ln K(\{\hat{Q}_l\}, \{\hat{P}_{l'}\}) + \sum_{l=1}^p \ln \rho(\hat{Q}_l) \right) \Big|_{\hat{Q}=\hat{Q}^*} \quad l < l' \quad (55)$$

The explicit form of  $K(\{\hat{Q}_l\}, \{\hat{P}_{l'}\})$  can be worked out similarly to Eq. (46) (see Eq.(40) and (78) of [9]) and one finds the 2nd equation of Eq. (55) yields  $\hat{P}_{l,l'} = 0$  meaning that the cross-correlation between spin fluctuations at different sites vanish (see Eq.(62) and (63) of [9]). Thus we can use the factorized form for the Jacobian.

Now inspecting Eq. (51), it is evident that a sensible choice for the scaling of the connectivity to obtain nontrivial result in  $M \rightarrow \infty$  limit is

$$c = \alpha M \quad (56)$$

parametrized by  $\alpha > 0$ . Then using Eq. (51) and Eq. (56) the saddle point equation Eq. (52) becomes

$$0 = \frac{\delta}{\delta \hat{Q}} \left( \frac{1}{2} \ln \det \hat{Q} + \alpha f_n(\hat{Q}, \hat{Q}^*, \dots, \hat{Q}^*) \right) \Big|_{\hat{Q}=\hat{Q}^*} = \frac{\delta}{\delta \hat{Q}} \left( \frac{1}{2} \ln \det \hat{Q} + \frac{\alpha}{p} f_n(\hat{Q}, \hat{Q}, \dots, \hat{Q}) \right) \Big|_{\hat{Q}=\hat{Q}^*} \quad (57)$$

The value of the Lagrange multiplier  $\lambda$  is fixed by Eq. (50), which requires vanishing of  $O(M)$  terms in  $\ln \rho(\hat{Q}^*) + \ln J(\hat{Q}^*)$ ,

$$0 = \lambda - 1 + M \left[ \frac{1}{2} n \ln(2\pi e) - \frac{1}{2} n \ln M + \frac{1}{2} \ln \det \hat{Q}^* + \alpha f_n(\hat{Q}^*, \hat{Q}^*, \dots, \hat{Q}^*) \right] \quad (58)$$

where we used Eq. (47), Eq. (51) and dropped subleading terms. Using this result together with Eq. (49), we find the saddle point value of the variational free-energy Eq. (43) in  $M \rightarrow \infty$  limit as,

$$-\beta \frac{\mathcal{F}_n[\rho(\hat{Q}^*)]}{N} = -\ln \rho(\hat{Q}^*) + M \frac{\alpha}{p} f_n(\hat{Q}^*, \hat{Q}^*, \dots, \hat{Q}^*) = 1 - \lambda - M \frac{\alpha}{p} (p-1) f_n(\hat{Q}^*, \hat{Q}^*, \dots, \hat{Q}^*) \\ = M \left[ \frac{1}{2} n \ln \left( \frac{2\pi e}{M} \right) + \frac{1}{2} \ln \det \hat{Q}^* + \frac{\alpha}{p} f_n(\hat{Q}^*, \hat{Q}^*, \dots, \hat{Q}^*) \right] \quad (59)$$

Note that if we regard  $\hat{Q}^*$  in the above expression as a variational parameter, we find a variational equation which is exactly the same as Eq. (57).

Next let us examine the interaction part of the free-energy to extract the explicit form of the replicated Mayer function. The interaction part of the free-energy of the replicated system reads as (see Eq. (38), Eq. (44)),

$$\begin{aligned}
& \frac{c}{p} \int_S d\bar{\mathbf{S}}_1 d\bar{\mathbf{S}}_2 \cdots d\bar{\mathbf{S}}_p \rho(\bar{\mathbf{S}}_1) \rho(\bar{\mathbf{S}}_2) \cdots \rho(\bar{\mathbf{S}}_p) f_n(\bar{\mathbf{S}}_1, \bar{\mathbf{S}}_2, \dots, \bar{\mathbf{S}}_p) \\
&= \frac{c}{p} \left[ -1 + \int \prod_{l=1}^p \{d\hat{Q}_l J(\hat{Q}_l) \rho(\hat{Q}_l)\} \left\langle \prod_{a=1}^n \exp \left( -\beta V \left( \delta - \frac{1}{\sqrt{M}} \sum_{\mu=1}^M \prod_{l=1}^p (S_l^a)^\mu \right) \right) \right\rangle_{\hat{Q}} \right] \\
&= \frac{c}{p} \left[ -1 + \int \prod_{l=1}^p \{d\hat{Q}_l J(\hat{Q}_l) \rho(\hat{Q}_l)\} \int \prod_{a=1}^n \left\{ \frac{d\kappa_a}{2\pi} e^{i\kappa_a \delta} \mathcal{Z}_{\kappa_a} \right\} \left\langle \exp \left( \sum_{a=1}^n \frac{-i\kappa_a}{\sqrt{M}} \sum_{\mu=1}^M \prod_{l=1}^p (S_l^a)^\mu \right) \right\rangle_{\hat{Q}} \right] \quad (60)
\end{aligned}$$

where we introduced a Fourier transform,

$$Z_\kappa \equiv \int dx e^{-i\kappa x} e^{-\beta V(x)} \quad (61)$$

and a short hand notation,

$$\langle \cdots \rangle_{\hat{Q}} \equiv \int \prod_{l=1}^p \left\{ d\bar{\mathbf{S}}_l \frac{1}{J(\hat{Q}_l)} \prod_{a \leq b} \delta \left( (Q_l)_{ab} - \frac{1}{M} \sum_{\mu=1}^M (S_l^a)^\mu (S_l^b)^\mu \right) \right\} \cdots \quad (62)$$

In the last equation  $Q_{aa} = 1$  (Eq. (45)) to enforce the normalization of the spins  $|(\mathbf{S}^a)^2| = M$  and  $\int d\bar{\mathbf{S}}$  is an unconstrained integral.

For  $M \gg 1$  we can evaluate the last factor in Eq. (60) by performing  $1/\sqrt{M}$  expansion,

$$\begin{aligned}
& \ln \left\langle \exp \left( \sum_{a=1}^n \frac{-i\kappa_a}{\sqrt{M}} \sum_{\mu=1}^M \prod_{l=1}^p (S_l^a)^\mu \right) \right\rangle_{\hat{Q}} \\
&= \ln \left[ 1 + \sum_{a=1}^n \frac{-i\kappa_a}{\sqrt{M}} \sum_{\mu=1}^M \prod_{l=1}^p \langle (S_l^a)^\mu \rangle_{\hat{Q}} + \frac{1}{2} \sum_{a=1}^n \sum_{b=1}^n (-i\kappa_a)(-i\kappa_b) \frac{1}{M} \prod_{l=1}^p \sum_{\mu, \nu=1}^M \langle (S_l^a)^\mu (S_l^b)^\nu \rangle_{\hat{Q}} + \cdots \right] \\
&\xrightarrow{M \rightarrow \infty} \frac{1}{2} \sum_{a,b=1}^n (-i\kappa_a)(-i\kappa_b) \prod_{l=1}^p (Q_l)_{ab} = \frac{1}{2} \sum_{a,b=1}^n (-i\kappa_a)(-i\kappa_b) \prod_{l=1}^p (Q_l)_{ab} \quad (63)
\end{aligned}$$

Here we used the fact that in the  $M \rightarrow \infty$  limit, different components of the spins  $S^\mu$  become independent from each other. This can be checked by introducing integral representation of the  $\delta$  function in Eq. (62) which can be evaluated by the saddle point method in  $M \rightarrow \infty$  limit.

To sum up we find the replicated Mayer function in the  $M \rightarrow \infty$  limit as,

$$\begin{aligned}
f_n(\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_p) &= -1 + \int \prod_{a=1}^n \left\{ \frac{d\kappa_a}{2\pi} e^{i\kappa_a \delta} \right\} \exp \left( \frac{1}{2} \sum_{a,b=1}^n \prod_{l=1}^p (Q_l)_{ab} (-i\kappa_a)(-i\kappa_b) \right) \prod_{a=1}^n \int dh_a e^{-i\kappa_a h_a} e^{-\beta V(h_a)} \\
&= -1 + \int \prod_{a=1}^n dh_a \left\{ \exp \left( \frac{1}{2} \sum_{a,b=1}^n \prod_{l=1}^p (Q_l)_{ab} \frac{\partial^2}{\partial h_a \partial h_b} \right) \prod_{a=1}^n \delta(\delta - h_a) \right\} \prod_a e^{-\beta V(h_a)} \\
&= -1 + \exp \left( \frac{1}{2} \sum_{a,b=1}^n \prod_{l=1}^p (Q_l)_{ab} \frac{\partial^2}{\partial h_a \partial h_b} \right) \prod_{a=1}^n e^{-\beta V(\delta + h_a)} \Big|_{\{h_a=0\}} \quad (64)
\end{aligned}$$

The last equation is obtained by repeating integrations by parts.

Collecting the above results we find the thermodynamic free-energy Eq. (37) as,

$$-\beta \frac{F}{NM} = -\beta f[\hat{Q}^*] \quad (65)$$

with the variational free-energy (more precisely free-entropy),

$$\begin{aligned}
-\beta f[\hat{Q}] &= \partial_n s_n[\hat{Q}] \Big|_{n=0} \\
s_n[\hat{Q}] &\equiv \frac{1}{2} \ln \det \hat{Q} - \frac{\alpha}{p} \mathcal{F}_{\text{int}}[\hat{Q}] \\
-\mathcal{F}_{\text{int}}[\hat{Q}] &\equiv \exp \left( \frac{1}{2} \sum_{a,b=1}^n (Q_{ab})^p \frac{\partial^2}{\partial h_a \partial h_b} \right) \prod_{a=1}^n e^{-\beta V(\delta + h_a)} \Big|_{\{h_a=0\}}
\end{aligned} \tag{66}$$

where we dropped off irrelevant constants. The saddle point  $\hat{Q}^*$  is determined by,

$$\left. \frac{\delta s[\hat{Q}]}{\delta Q_{ab}} \right|_{\hat{Q}=\hat{Q}^*} = 0 \tag{67}$$

for all  $a \neq b$ .

The pressure Eq. (23) can be computed using Eq. (56), Eq. (65) and Eq. (66) as,

$$\Pi = -\frac{p}{\alpha} \frac{\partial \beta f[\hat{Q}^*]}{\partial \delta} = -\frac{\partial}{\partial \delta} \partial_n \mathcal{F}_{\text{int}} \Big|_{n=0}. \tag{68}$$

and similarly the distribution function of the gap Eq. (25) as,

$$g(r) = -\frac{p}{\alpha} \frac{\delta \beta f[\hat{Q}^*]}{\delta(-\beta V(r))} = -\frac{\delta}{\delta(-\beta V(r))} \partial_n \mathcal{F}_{\text{int}} \Big|_{n=0}. \tag{69}$$

#### 4. Gaussian ansatz

Finally let us note that one can check that the above result can be reproduced by assuming an Gaussian ansatz for the replicated spin density,

$$\rho_{\text{Gaussian}}(\mathbf{S}) = \frac{e^{-\frac{1}{2} \sum_{a,b=1}^n (Q^{-1})_{ab} \sum_{\mu=1}^M (S^a)^\mu (S^b)^\mu}}{\sqrt{2\pi(\det \hat{Q})^M}} \tag{70}$$

The situation is essentially the same as that of hardspheres in large dimensional limit [9–11].

### III. REPLICA SYMMETRIC (RS) ANSATZ AND THE LIQUID PHASE

#### A. Formulation

In the replica symmetric (RS) ansatz we assume the following form of the overlap matrix parametrized by a single parameter  $q$ ,

$$Q_{ab}^{\text{RS}} = (1 - q)\delta_{ab} + q. \tag{71}$$

##### 1. Free-energy

First let us compute the free-energy within the RS ansatz. Using Eq. (71) we find,

$$\ln \det \hat{Q}^{\text{RS}} = \ln[1 + (n - 1)q] + (n - 1) \ln(1 - q) \tag{72}$$

so that the entropic part of the free-energy is obtained as

$$\left. \frac{1}{2} \partial_n \ln \det \hat{Q}^{\text{RS}} \right|_{n=0} = \frac{1}{2} \left( \frac{q}{1 - q} + \ln(1 - q) \right). \tag{73}$$



The interaction part of the free-energy is obtained as,

$$\begin{aligned}
-\mathcal{F}_{\text{int}}[\hat{Q}^{\text{RS}}] &= \exp \left( \frac{1}{2} \sum_{a,b=1}^n [(1-q^p)\delta_{ab} + q^p] \frac{\partial^2}{\partial h_a \partial h_b} \right) \prod_{a=1}^n e^{-\beta V(\delta+h_a)} \Bigg|_{\{h_a=0\}} \\
&= \exp \left( \frac{1}{2} q^p \sum_{a,b=1}^n \frac{\partial^2}{\partial h_a \partial h_b} \right) \prod_{a=1}^n \left\{ \exp \left( \frac{1}{2} (1-q^p) \frac{\partial^2}{\partial h_a^2} \right) e^{-\beta V(\delta+h_a)} \right\} \Bigg|_{h_a=0} \\
&= \gamma_{q^p} \otimes (\gamma_{1-q^p} \otimes e^{-n\beta V(\delta)})
\end{aligned} \tag{74}$$

where we used the formula

$$\exp \left( \frac{a}{2} \frac{\partial^2}{\partial h^2} \right) A(h) = \gamma_a \otimes A(h) \tag{75}$$

and the following short hand notations:  $\gamma_a(x)$  is a Gaussian with zero mean and variance  $a$ ,

$$\gamma_a(x) = \frac{1}{\sqrt{2\pi a}} e^{-\frac{x^2}{2a}}, \tag{76}$$

by which we write a convolution of a function  $A(x)$  with the Gaussian as,

$$\gamma_a \otimes A(x) \equiv \int dy \frac{e^{-\frac{y^2}{2a}}}{\sqrt{2\pi a}} A(x-y) = \int \mathcal{D}z A(x - \sqrt{a}z) \tag{77}$$

where

$$\int \mathcal{D}z \dots \equiv \int dz \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \dots \tag{78}$$

Collecting the above results we obtain the variational free-entropy Eq. (66) within the RS ansatz as

$$-\beta f_{\text{RS}}(q) = \partial_n s_{\text{RS}}(q)|_{n=0} = \frac{1}{2} \left( \frac{q}{1-q} + \ln(1-q) \right) + \frac{\alpha}{p} \int \mathcal{D}z_0 \ln \int \mathcal{D}z_1 e^{-\beta V(\delta - \sqrt{1-q^p}z_1 - \sqrt{q^p}z_0)} \tag{79}$$

## 2. The saddle point equation

The saddle point equation for the order parameter  $q$  is obtained as,

$$\begin{aligned}
0 &= \frac{\partial(-\beta f_{\text{RS}}(q))}{\partial q} = \frac{1}{2} \frac{q}{(1-q)^2} - \frac{\alpha}{p} \frac{pq^{p-1}}{2} \int \mathcal{D}z_0 \left( \frac{\int \mathcal{D}z_1 (e^{-\beta V(x)})'}{\int \mathcal{D}z_1 e^{-\beta V(x)}} \right)^2 \Bigg|_{x=\delta - \sqrt{1-q^p}z_1 - \sqrt{q^p}z_0} \\
&= \frac{1}{2} \frac{q}{(1-q)^2} \mathcal{G}(q)
\end{aligned} \tag{80}$$

where we introduced

$$\mathcal{G}(q) \equiv 1 - \alpha(1-q)^2 q^{p-2} \int \mathcal{D}z_0 \left( \frac{\int \mathcal{D}z_1 (e^{-\beta V(x)})'}{\int \mathcal{D}z_1 e^{-\beta V(x)}} \right)^2 \Bigg|_{x=\delta - \sqrt{1-q^p}z_1 - \sqrt{q^p}z_0} \tag{81}$$

## 3. Pressure and distribution of gap

Using Eq. (79) we obtain the pressure Eq. (68) as,

$$\Pi = \int \mathcal{D}z_0 \frac{\int \mathcal{D}z_1 (e^{-\beta V(x)})'}{\int \mathcal{D}z_1 e^{-\beta V(x)}} \Bigg|_{x=\delta - \sqrt{1-q^p}z_1 - \sqrt{q^p}z_0} \tag{82}$$

and similarly the distribution of the gap Eq. (69) as,

$$\begin{aligned}
g(r) &= \int \mathcal{D}z_0 \frac{\int \mathcal{D}z_1 \delta(x-r) e^{-\beta V(x)} \Big|_{x=\delta-\sqrt{1-q^p}z_1-\sqrt{q^p}z_0}}{\int \mathcal{D}z_1 e^{-\beta V(x)} \Big|_{x=\delta-\sqrt{1-q^p}z_1-\sqrt{q^p}z_0}} \\
&= e^{-\beta V(r)} \int \mathcal{D}z_0 \frac{\gamma_{1-q^p}(\delta-\sqrt{q^p}z_0-r)}{\int \mathcal{D}z_1 e^{-\beta V(x)} \Big|_{x=\delta-\sqrt{1-q^p}z_1-\sqrt{q^p}z_0}}
\end{aligned} \tag{83}$$

which is properly normalized such that  $\int dr g(r) = 1$ . One can also check easily that the 'virial equation' Eq. (26) for the pressure is satisfied, as it should be.

### B. The liquid phase : $q = 0$ solution

Apparently  $q = 0$  representing the liquid state is always a solution of the RS saddle point equation Eq. (80) for  $p \geq 2$ . The stability of the solution must be examined by studying the eigenvalues of the Hessian matrix reported in the appendix A.

#### 1. $p = 2$ case

From the results in appendix A we find for the  $q = 0$  solution for  $p = 2$ ,

$$M_1 = 2 - 2\alpha \left( \frac{\gamma_1 \otimes (e^{-\beta V(\delta)})'}{\gamma_1 \otimes e^{-\beta V(\delta)}} \right)^2 \tag{84}$$

$$M_2 = M_3 = 0 \tag{85}$$

from which we find the eigenvalues of the Hessian matrix as,

$$\lambda_R = \lambda_L = \lambda_A = 2 - 2\alpha \left( \frac{\gamma_1 \otimes (e^{-\beta V(\delta)})'}{\gamma_1 \otimes e^{-\beta V(\delta)}} \right)^2 \tag{86}$$

which vanishes at,

$$\alpha_c(\delta) = \left( \frac{\gamma_1 \otimes (e^{-\beta V(\delta)})'}{\gamma_1 \otimes e^{-\beta V(\delta)}} \right)^{-2} \tag{87}$$

For  $\alpha < \alpha_c(\delta)$ , the eigenvalues are positive so that the  $q = 0$  solution is stable but it becomes unstable for  $\alpha > \alpha_c(\delta)$ .

Interestingly we see that at the critical point  $\alpha = \alpha_c(\delta)$ ,  $q = 0$  solves also  $\mathcal{G}(q) = 0$  (see Eq. (81)). Since  $q \neq 0$  solution must solve  $\mathcal{G}(q) = 0$ , this suggests a possibility of continuous phase transition at the critical point such that  $q \neq 0$  solution emerges continuously.

#### 2. $p > 2$ case

For  $p > 2$ , using the results reported in appendix A, we find  $\lambda_R = \lambda_L = \lambda_A = 2 > 0$  so that  $q = 0$  solution is always stable. Thus contrary to the  $p = 2$  model, we find the liquid phase described by the  $q = 0$  RS solution is always (meta)stable. The situation is very similar to the  $p$ -spin spherical spinglass models [21]. Then we are naturally lead to consider the possibility of a glass phase represented by 1step replica symmetry breaking (1RSB) much as in the  $p$ -spin SG models.

### C. Results on the hardcore model

To investigate further we now we focus on the hardcore potential Eq. (22). For the hardcore potential we find the RS free-energy Eq. (79) as,

$$-\beta f_{\text{RS}}(q) = \frac{1}{2} \left( \frac{q}{1-q} + \ln(1-q) \right) + \frac{\alpha}{p} \int \mathcal{D}z_0 \ln \Theta \left( \frac{\delta - \sqrt{q^p} z}{\sqrt{2(1-q^p)}} \right) \tag{88}$$

where we introduced a function  $\Theta(x)$

$$\Theta(x) \equiv \int_{-\infty}^x \frac{dz}{\sqrt{\pi}} e^{-z^2} = \gamma_{1/2} \otimes \theta(x) = \frac{1}{2}(1 + \text{erf}(x)), \quad (89)$$

with  $\text{erf}(x)$  being the error function,

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dy e^{-y^2} = -\text{erf}(-x), \quad (90)$$

which behaves for  $x \rightarrow \infty$  as,

$$\text{erf}(x) = 1 - \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x} \left( 1 - \frac{1}{2x^2} + \frac{3}{(2x^2)^2} + \dots \right). \quad (91)$$

The function  $\mathcal{G}(q)$  defined in Eq. (81) becomes,

$$\mathcal{G}(q) = 1 - \alpha(1-q)^2 q^{p-2} \frac{1}{2(1-q^p)} \int \mathcal{D}z_0 r^2(x) \Big|_{x=\frac{\delta-\sqrt{q^p}z_0}{\sqrt{2(1-q^p)}}} \quad (92)$$

where we introduced,

$$r(x) \equiv \frac{\Theta'(x)}{\Theta(x)} = \frac{e^{-x^2}}{\sqrt{\pi}} / \Theta(x) \quad (93)$$

which behaves asymptotically as,

$$r(x) \simeq \begin{cases} -2x \left( 1 - \frac{1}{2x^2} + \frac{3}{(2x^2)^2} + \dots \right)^{-1} & x \rightarrow -\infty \\ 0 & x \rightarrow \infty \end{cases} \quad (94)$$

#### 1. $q = 0$ RS solution and its stability

Within the liquid state  $q = 0$ , the pressure is obtained as,

$$\Pi = \frac{1}{\sqrt{2}} r(\delta/\sqrt{2}) = \frac{1}{\sqrt{2}} \frac{\Theta'(\delta/\sqrt{2})}{\Theta(\delta/\sqrt{2})} \quad (95)$$

As shown in the left panel of Fig. 6, the pressure monotonically increases by decreasing  $\delta$  as expected. We display in the right panel of Fig. 6 b) the behavior of the distribution of gap,

$$g(r) = \frac{\theta(r)}{\Theta(\delta/\sqrt{2})} \frac{e^{-\frac{(\delta-r)^2}{2}}}{\sqrt{2\pi}} \quad (96)$$

We see that the peak around  $r = 0$  develops by decreasing  $\delta$  as expected.

As we found above  $q = 0$  solution is always stable for  $p > 2$  body interactions. For the  $p = 2$  body hardcore interaction, we find the  $q = 0$  solution becomes unstable for  $\alpha > \alpha_c(\delta)$  with

$$\alpha_c(\delta) = 2r^{-2}(\delta/\sqrt{2}) = 2 \left( \frac{\Theta'(\delta/\sqrt{2})}{\Theta(\delta/\sqrt{2})} \right)^{-2}. \quad (97)$$

#### 2. $q \neq 0$ RS solution for the $p = 2$ body case and its stability

Now we are naturally led to examine the possibility of the  $q \neq 0$  solution in the case of  $p = 2$  within the RS ansatz. It must solve  $\mathcal{G}(q) = 0$  (see Eq. (80)), where the function  $\mathcal{G}(q)$  for the hardcore model is given by Eq. (92). Expanding  $\mathcal{G}(q)$  up to order  $O(q^2)$  we find,

$$0 = \mathcal{G}(q) = 1 - \frac{\alpha}{\alpha_c(\delta)} \left[ 1 - 2q + (2 - 2x_0 - r_0)q^2 \right] + O(q^4) \quad (98)$$

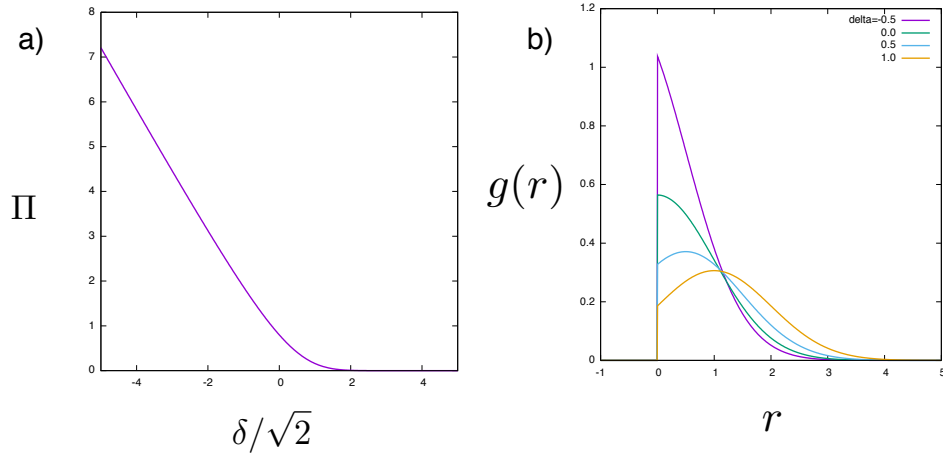


FIG. 6. Liquid phase of the hardcore model: a) behavior of the pressure  $\Pi$  and b) the distribution of the gap  $g(r)$ .

where

$$x_0 \equiv \frac{\delta}{\sqrt{2}} \quad r_0 \equiv r(x_0) = \frac{\Theta'(x_0)}{\Theta(x_0)} \quad (99)$$

The above equation can be solved for  $q$  to find,

$$q = \frac{1}{2}\epsilon - \frac{1}{4} \left( 1 + x_0 + \frac{r_0}{2} \right) \epsilon^2 + O(\epsilon^3) \quad \epsilon \equiv \frac{\alpha}{\alpha_c(\delta)} - 1 \quad (100)$$

Thus we find that  $q \neq 0$  solution emerges at the critical point  $\alpha = \alpha_c(\delta)$ , where the  $q = 0$  solution becomes unstable, and the EA order parameter  $q$  grows continuously increasing  $\alpha$ .

Now let us examine the stability of this solution. From the result reported in appendix A we find the replicon eigenvalue  $\lambda_R$  as,

$$\lambda_R = \frac{2}{(1-q)^2} \mathcal{R}(q) \quad (101)$$

$$\mathcal{R}(q) = 1 - \frac{\alpha}{2(1+q)^2} \int \mathcal{D}z_0 \left( r^2(x)(1-q^2) + 4q^2 x^2 r^2(x) + x r^3(x) + \frac{r^4(x)}{4} \right)_{x=\frac{\delta - \sqrt{qP} z_0}{\sqrt{2(1-q^P)}}} \quad (102)$$

$$= 1 - \frac{\alpha}{\alpha_c(\delta)} \left[ 1 - 2q + \frac{1}{2}(5r_0^2 + 16r_0 x_0 + 12x_0^2 + 2)q^2 + O(q^3) \right] \quad (103)$$

In the last equation we made an expansion in series of  $q$  which can be obtained by using  $r'(x) = -2xr(x) - r^2(x)$  which follows from Eq. (93). Comparing the function  $\mathcal{G}(q)$  and  $\mathcal{R}(q)$  we notice that they are identical up to  $O(q)$  but different in the  $O(q^2)$  terms. Using Eq. (100) in Eq. (103) we find up to  $O(\epsilon^2)$ ,

$$\lambda_R = \frac{2}{(1-q)^2} A(x_0) \epsilon^2 \quad A(x) = \frac{1}{4} - \frac{x}{2} - \frac{r(x)}{4} - \frac{5}{8} r^2(x) - 2xr(x) - \frac{3}{2} x^2 \quad (104)$$

It turns out that  $A(x)$  is a monotonically decreasing function of  $x$ . Using the asymptotic behavior of the function  $r(x)$  given in Eq. (94) one can find  $\lim_{x \rightarrow -\infty} A(x) = -1/4$ . Thus we find that the replicon eigenvalue is definitely negative meaning that the RS solution is unstable for  $\alpha > \alpha_c(\delta)$ . We also checked numerically, solving  $\mathcal{G}(q) = 0$  for  $q$  and evaluating  $\lambda_R$  (Eq. (102)), we checked this is indeed the case in the whole regime of  $\alpha > \alpha_c(\delta)$ . Thus the replica symmetry must be broken for  $\alpha > \alpha_c(\delta)$ .

Remarkably the situation is very similar to the Sherrington-Kirkpatrick (SK) model for the spin glasses [27, 35, 36]. To summarize we find the liquid solution described by the  $q = 0$  RS solution which becomes unstable approaching the critical point  $\alpha_c(\delta)$  where all eigenvalues of the Hessian matrix vanish. The line  $\alpha = \alpha_c(\delta)$  is the equivalent of the d'Almeida-Thouless (AT) line [27]. It immediately means divergence of the so called spin-glass susceptibility negative divergence of non-linear compressibility  $d^2 p / d\delta^2$  much as the spinglass transition of the SK model [20, 37]. Beyond the transition point, going into the glass phase, we have to consider breaking of the replica symmetry.[38–40].

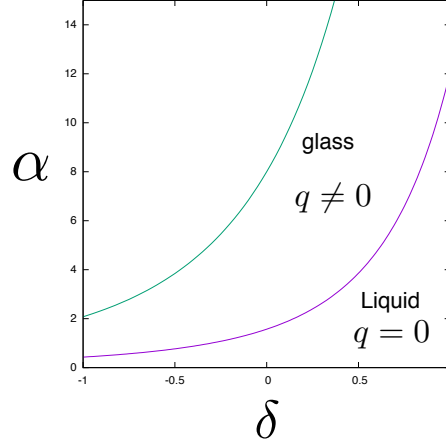


FIG. 7. The phase diagram of the  $p = 2$  body hardcore model within the replica symmetric (RS) ansatz:  $q > 0$  RS solution emerges continuously at the lower curve which represents  $\alpha = \alpha_c(\delta)$  Eq. (97). The value of the order parameter saturates  $q \rightarrow 1$  approaching the upper curve  $\alpha = \alpha_j(\delta)$  Eq. (106), which is the jamming line within the RS ansatz. The lower curve coincides with the AT line above which the RS solution becomes unstable.

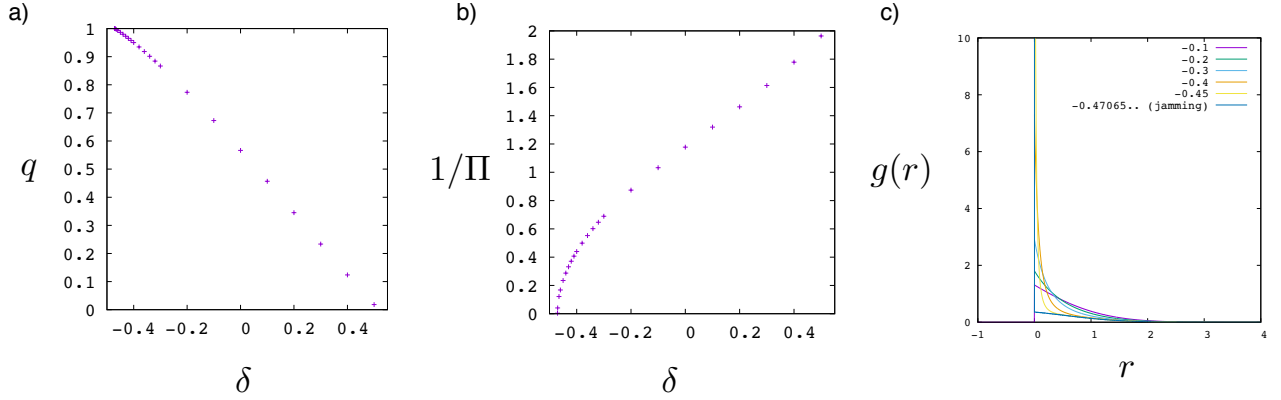


FIG. 8. Glass phase of the hardcore model within the RS ansatz (here we choose  $\alpha = 4$ ): a) behavior of the order parameter  $q$ , b) inverse of the pressure  $\Pi$ , c) the distribution of the gap  $g(r)$

### 3. Glass phase and jamming within the RS ansatz for the $p = 2$ body case

Here we study the glass phase  $p = 2$  hardcore model within the RS ansatz. In Fig. 8, we display an example of a set of solutions of the RS saddle point equation Eq. (80) with Eq. (92) for a  $\alpha = 4$  with varying  $\delta$ . As shown in the panel a), the glass order parameter  $q$  emerges continuously at the critical point  $\delta_c \sim 0.51$  (determined by  $\alpha_c(\delta_c) = 4$ , see Fig. 7) and increases by decreasing  $\delta$  and saturates to  $q = 1$  approaching the jamming point.

The location of the jamming point can be analyzed as follows. We find  $\mathcal{G}(q)$  given in Eq. (92) for  $p = 2$  becomes in the limit  $q \rightarrow 1$ ,

$$\lim_{q \rightarrow 1} \mathcal{G}(q) = 1 - \alpha \frac{\alpha}{4} \lim_{q \rightarrow 1} (1 - q) \int \mathcal{D}z_0 r^2(x) \Big|_{x=\frac{\delta-z_0}{2\sqrt{1-q}}} = 1 - \frac{\alpha}{4} \int_0^\infty \frac{dy}{\sqrt{2\pi}} e^{-(\delta+y)^2/2} y^2 \quad (105)$$

In the last equation we used the asymptotic behavior of the function  $r(x)$  given in Eq. (94). Thus we find the jamming line  $\alpha = \alpha_j(\delta)$ ,

$$\alpha_j(\delta) = \frac{4}{\int_0^\infty \frac{dy}{\sqrt{2\pi}} e^{-(\delta+y)^2/2} y^2} \quad (106)$$

which is also displayed in Fig. 7.

The pressure Eq. (82) becomes for the hardcore model,

$$\Pi = \frac{1}{\sqrt{2}} \int \mathcal{D}z_0 \, r(x) \Big|_{x=\frac{\delta-\sqrt{q^p}z_0}{\sqrt{2(1-q^p)}}} \xrightarrow{q \rightarrow 1} \int_{\delta}^{\infty} \mathcal{D}z_0 \frac{(z_0 - \delta)}{\sqrt{1-q^p}} \propto \frac{1}{\sqrt{1-q}} \quad (107)$$

where we used the asymptotic expansion Eq. (94). Thus as expected the pressure diverges by jamming (see Fig. 8 b)).

Next let us examine the distribution of the gap Eq. (83) which becomes for the hardcore model,

$$g(r) = \theta(r) \int \mathcal{D}z_0 \frac{\gamma_{1-q^p}(\delta - \sqrt{q^p}z_0 - r)}{\Theta\left(\frac{\delta - \sqrt{q^p}z_0}{\sqrt{2(1-q^p)}}\right)}. \quad (108)$$

Let us consider the behavior in the jamming limit  $q \rightarrow 1$  in the following two ways (see Fig. 8 c)).

1. For *fixed* finite  $r$ , sending  $q \rightarrow 1$ , we find

$$\lim_{q \rightarrow 1} g(r) = \theta(r) \frac{e^{-\frac{(\delta-r)^2}{2}}}{\sqrt{2\pi}} \quad (109)$$

This is because  $\gamma_{1-q^p}(\delta - \sqrt{q^p}z_0 - r)$  becomes a delta function in  $q \rightarrow 1$  limit and  $\lim_{X \rightarrow \infty} \Theta(X) = 1$ .

2. In the vanishing region around  $r = 0$  parametrized as  $r = (1 - q^p)\lambda$  we find a different behavior as follows. Assuming  $q \sim 1$  we find for  $r > 0$ ,

$$\begin{aligned} g(r) &\sim \int_{\delta}^{\infty} \frac{dz_0}{\sqrt{2\pi}} \frac{e^{-\frac{(\delta-z_0-r)^2}{2(1-q^p)}}}{2\pi(\sqrt{1-q^p})} 2\sqrt{\pi}|X|e^{X^2} \Big|_{X=\frac{\delta-\sqrt{q^p}z_0}{2\sqrt{1-q^p}}} \\ &\xrightarrow{q \rightarrow 1, \text{fixed } \lambda} \frac{1}{1-q^p} \int_0^{\infty} \frac{dy}{\sqrt{2\pi}} e^{-\frac{(\delta+y)^2}{2}} y e^{-\lambda y} \quad \lambda = \frac{r}{1-q^p} \end{aligned} \quad (110)$$

In the 1st equation we dropped contribution from  $\int_{-\infty}^{\delta} dz_0..$  which can be neglected compared with the contribution from  $\int_{\delta}^{\infty} dz_0..$  and used the asymptotic behavior of the error function Eq. (91) which implies  $\Theta(-X) \sim \frac{e^{-X^2}}{2\sqrt{\pi}X}$  for  $X \gg 1$ . Thus in the jamming limit  $q \rightarrow 1$  we find diverging peak in the “contact region” around  $r = 0$  whose height diverging as  $1/(1-q)$  and the width vanishing as  $1-q$ .

## IV. REPLICA SYMMETRY BREAKING (RSB) ANSATZ

### A. Parisi’s ansatz

We now study solutions with replica symmetry breaking using the Parisi’ ansatz [39]: we assume the following structure of the glass order parameter in the glass phase.

$$Q_{ab}^{k\text{-RSB}} = q_0 + \sum_{i=1}^{k+1} (q_i - q_{i-1}) I_{ab}^{m_i} = \sum_{i=0}^{k+1} q_i (I_{ab}^{m_i} - I_{ab}^{m_{i+1}}) \quad (111)$$

where  $I_{ab}^m$  is a kind of generalized (‘fat’) identity matrix of size  $n \times n$  composed of blocks of size  $m \times m$ . (see Fig. 9) The matrix elements in the diagonal blocks, are all 1 while those in the off-diagonal blocks are all 0. The Parisi’s matrix has a hierarchical structure such that

$$1 = m_{k+1} < m_k < \dots < m_1 < m_1 < m_0 = n \quad (112)$$

which becomes

$$0 = m_0 < m_1 < \dots < m_k < m_{k+1} = 1 \quad (113)$$

in the  $n \rightarrow 0$  limit. The expression Eq. (111) is valid also for the diagonal part by introducing

$$q_{k+1} = 1, \quad (114)$$

which reflects the normalization condition Eq. (45).

Let us note that we may sometimes extend the labels  $m$ ’s Eq. (113) introducing an additional label  $m_{k+2}$  just for conveniences when we deal with recursion formulas.



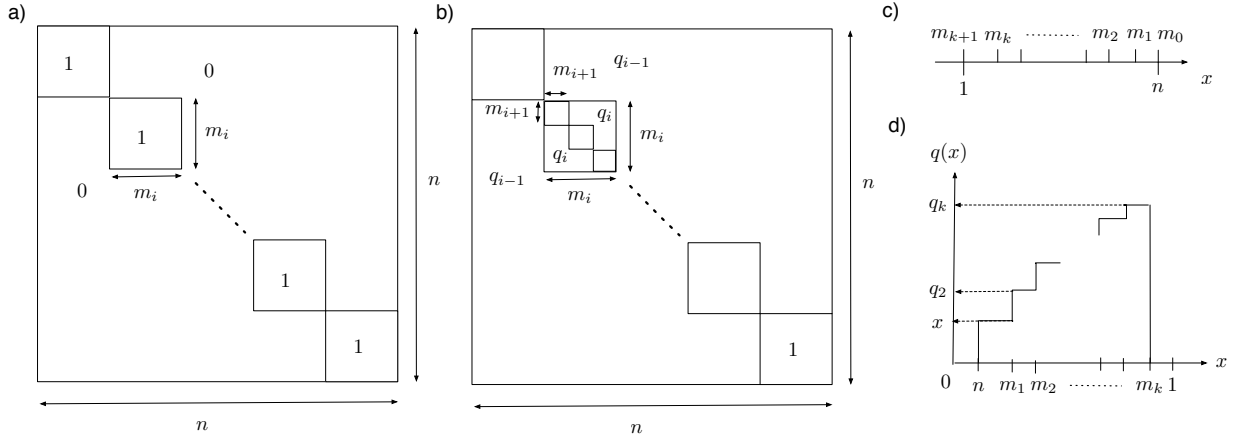


FIG. 9. Parametrization of the Parisi's matrix a) the 'fat' identity matrix  $I_{ab}^{m_i}$  b) Parisi's order parameter matrix Eq. (111) c) the hierarchy of the sizes  $m_i$  of the sub-matrices d) the  $q(x)$  function with  $0 < n < 1$ .

## B. Free-energy

Let us evaluate the free-energy Eq. (65)-Eq. (66) using the above ansatz. To compute the entropic part of the free-energy one needs to evaluate  $\ln \det \hat{Q}^{k\text{-RSB}}$ . Given the hierarchical structure of the Parisi's matrix, one can obtain them in a recursive fashion and one finds [41],

$$\ln \det \hat{Q}^{k\text{-RSB}} = \ln \left( 1 + \sum_{j=0}^k (m_j - m_{j+1}) q_j \right) \quad (115)$$

$$+ n \sum_{i=0}^k \left( \frac{1}{m_{i+1}} - \frac{1}{m_i} \right) \ln \left( 1 + \sum_{j=i}^k (m_j - m_{j+1}) q_j - m_i q_i \right) \quad (116)$$

Remembering that  $m_0 = n$  we find,

$$\partial_n \ln \det \hat{Q}^{k\text{-RSB}} \Big|_{n=0} = \frac{q_0}{G_0} + \frac{1}{m_1} \ln G_0 + \sum_{i=1}^k \left( \frac{1}{m_{i+1}} - \frac{1}{m_i} \right) \ln G_i \quad (117)$$

with

$$G_i = 1 + \sum_{j=i}^k (m_j - m_{j+1}) q_j - m_i q_i \quad i = 0, 1, \dots, k \quad (118)$$

which implies

$$q_i = 1 - G_k + \sum_{j=i+1}^k \frac{1}{m_j} (G_j - G_{j-1}) \quad i = 0, 1, \dots, k \quad (119)$$

The interaction part of free-energy can also be evaluated in a recursive fashion [42]. One finds,

$$\begin{aligned} -\mathcal{F}_{\text{int}}[\hat{Q}^{k\text{-RSB}}] &= \prod_{l=0}^{k+1} \exp \left( \frac{\Lambda_l}{2} \sum_{a,b=1}^n I_{ab}^{m_l} \frac{\partial^2}{\partial h_a \partial h_b} \right) \prod_{a=1}^n e^{-\beta V(\delta + h_a)} \Big|_{\{h_a=0\}} \\ &= \prod_{l=0}^k \exp \left( \frac{\Lambda_l}{2} \sum_{a,b=1}^n I_{ab}^{m_l} \frac{\partial^2}{\partial h_a \partial h_b} \right) \prod_{a=1}^n g(m_{k+1}, h_a) \Big|_{\{h_a=0\}} \end{aligned} \quad (120)$$

where we introduced

$$\begin{aligned} \Lambda_0 &\equiv \lambda_0 \\ \Lambda_i &\equiv \lambda_i - \lambda_{i-1} \quad i = 1, 2, \dots, k+1 \end{aligned} \quad (121)$$

with

$$\lambda_i \equiv q_i^p. \quad (122)$$

In the 2nd equation we used  $I_{ab}^{m_{k+1}=1} = \delta_{ab}$  and introduced

$$g(m_{k+1}, h) \equiv \gamma_{\Lambda_{k+1}} \otimes e^{-\beta V(h)} = \int \mathcal{D}z_{k+1} e^{-\beta V(h - \sqrt{\Lambda_{k+1}} z_{k+1})} \quad (123)$$

In the last equation we used the formula Eq. (75).

The expression Eq. (120) naturally motivates a family of functions  $g$ 's which obey a recursion relation,

$$g(m_l, h) = e^{\frac{\Lambda_l}{2} \frac{\partial^2}{\partial h^2}} g^{\frac{m_l}{m_{l+1}}} (m_{l+1}, h) = \gamma_{\Lambda_l} \otimes g^{\frac{m_l}{m_{l+1}}} (m_{l+1}, h) \quad (124)$$

$$= \int \mathcal{D}z_l g^{\frac{m_l}{m_{l+1}}} (m_{l+1}, h - \sqrt{\Lambda_l} z_l) \quad (125)$$

for  $l = 0, 1, \dots, k$ . It is easy to see that

$$\begin{aligned} -\mathcal{F}_{\text{int}}[\hat{Q}^{k-\text{RSB}}] &= \gamma_{\Lambda_0} \otimes \left( \gamma_{\Lambda_1} \otimes (\dots \gamma_{\Lambda_k} \otimes g^{m_k}(m_{k+1}, \delta) \dots)^{m_1/m_2} \right)^{m_0/m_1} \\ &= \dots \\ &= \gamma_{\Lambda_0} \otimes \left( \gamma_{\Lambda_1} \otimes g^{m_1/m_2}(m_2, \delta) \right)^{m_0/m_1} \\ &= \gamma_{\Lambda_0} \otimes g^{m_0/m_1}(m_1, \delta) \\ &= g(m_0, \delta) \end{aligned} \quad (126)$$

Equivalently we can introduce a related family of functions  $f$ 's,

$$f(m, h) \equiv -\frac{1}{m} \ln g(m, h) \quad (127)$$

which follows a recursion relation,

$$e^{-m_i f(m_i, h)} = \gamma_{\Lambda_i} \otimes e^{-m_i f(m_{i+1}, h)} = \int \mathcal{D}z_i e^{-m_i f(m_{i+1}, h - \sqrt{\Lambda_i} z_i)} \quad (128)$$

for  $i = 0, 1, \dots, k+1$  with the boundary condition,

$$e^{-m_{k+1} f(m_{k+1}, h)} = \int \mathcal{D}z_{k+1} e^{-m_{k+1} \beta V(h - \sqrt{\Lambda_{k+1}} z_{k+1})}. \quad (129)$$

where  $m_{k+1} = 1$ . We may also express the boundary condition as,

$$f(m_{k+2}, h) = \beta V(h). \quad (130)$$

by introducing  $m_{k+2}$  just as an additional label for convenience. Remembering that  $m_0 = n$  we find the interaction part of the free-energy becomes,

$$\begin{aligned} -\partial_n \mathcal{F}_{\text{int}}[\hat{Q}^{k-\text{RSB}}] \Big|_{n=0} &= \partial_{m_0} g(m_0, \delta) \Big|_{m_0=n=0} = -f(m_0 = 0, \delta) \\ &= \gamma_{\Lambda_0} \otimes (-f(m_1, \delta)) = - \int \mathcal{D}z_0 f(m_1, \delta - \sqrt{\Lambda_0} z_0) \end{aligned} \quad (131)$$

Finally collecting the above results we obtain the free-energy within the  $k$ -RSB ansatz as,

$$\begin{aligned} -\beta f_{k-\text{RSB}}[\hat{Q}] &= \frac{1}{2} \frac{q_0}{G_0} + \frac{1}{2} \frac{1}{m_1} \ln G_0 + \frac{1}{2} \sum_{i=1}^k \left( \frac{1}{m_{i+1}} - \frac{1}{m_i} \right) \ln G_i \\ &\quad + \frac{\alpha}{p} \int \mathcal{D}z_0 (-f(m_1, \delta - \sqrt{\Lambda_0} z_0)) \end{aligned} \quad (132)$$

1.  $k = 0$  case: RS

Let us check if  $k = 0$  case recovers the result we obtained previously for the replica symmetric (RS) ansatz.

$$-\beta f_{0\text{-RSB}}(q_0) = \frac{1}{2} \frac{q_0}{(1 - q_0)} + \frac{1}{2} \ln(1 - q_0) + \frac{\alpha}{p} \int \mathcal{D}z_0 \ln \int \mathcal{D}z_1 e^{-\beta V(\Xi(\delta, q_0))} \quad (133)$$

with

$$\Xi(\delta, q_0) = \delta - \sqrt{1 - q_0^p} z_1 - \sqrt{q_0^p} z_0 \quad (134)$$

where we used  $G_0 = 1 - (m_1 - m_0)q_0$  and that  $m_0 = 0$  and  $m_1 = 1$ . In the 2nd equation we used Eq. (129). The result agrees with Eq. (79) as it should.

2.  $k = 1$  case: 1 RSB

For the  $k = 1$  RSB case we find,

$$\begin{aligned} -\beta f_{1\text{-RSB}}(q_0, q_1, m_1) &= \frac{1}{2} \frac{q_0}{1 - m_1 q_0 + (m_1 - 1)q_1} + \frac{1}{2} \frac{1}{m_1} \ln(1 - m_1 q_0 + (m_1 - 1)q_1) + \frac{1}{2} \left(1 - \frac{1}{m_1}\right) \ln(1 - q_1) \\ &+ \frac{\alpha}{p} \frac{1}{m_1} \int \mathcal{D}z_0 \ln \int \mathcal{D}z_1 \left[ \int \mathcal{D}z_2 e^{-\beta V(\Xi(\delta, q_0, q_1))} \right]^{m_1} \end{aligned} \quad (135)$$

with

$$\Xi(\delta, q_0, q_1) = \delta - \sqrt{q_0^p} z_0 - \sqrt{q_1^p - q_0^p} z_1 - \sqrt{1 - q_1^p} z_2. \quad (136)$$

where  $q_0$  and  $q_1$  must be determined through the saddle point equations which we discuss in sec. IV E).

An important quantity is the complexity or the configurational entropy  $\Sigma(f)$ , which describes the exponentially large number of states  $\propto e^{N\Sigma(f)} df$  with free-energy density between  $f$  and  $f + df$  in the glass phase. Using Monasson's prescription [43], one can construct the complexity function  $\Xi(f)$  using  $m = m_1$  as a parameter;

$$\Sigma^*(m) = m^2 \partial_m \beta f_{1\text{-RSB}}(q_0, q_1, m) \quad (137)$$

$$\beta f^*(m) = \partial_m (m \beta f_{1\text{-RSB}})(q_0, q_1, m) \quad (138)$$

Thus extremization of the free-energy with respect to  $m$ ,  $0 = \partial_m \beta f_{1\text{-RSB}}(q_0, q_1, m)$  amounts to force the complexity to vanish  $\Sigma^*(m)$  [43].

We readily find the following explicit expressions,

$$\begin{aligned} \beta f^*(m_1) &= -\frac{1}{2} \frac{q_0}{1 - m_1 q_0 + (m_1 - 1)q_1} - \frac{1}{2} \ln(1 - q_1) \\ &- \frac{1}{2} \left( -\frac{m_1 q_0}{[1 - m_1 q_0 + (m_1 - 1)q_1]^2} + \frac{1}{1 - m_1 q_0 + (m_1 - 1)q_1} \right) (q_1 - q_0) \\ &- \frac{\alpha}{p} \int \mathcal{D}z_0 \frac{\int \mathcal{D}z_1 \left[ \int \mathcal{D}z_2 e^{-\beta V(\Xi(\delta, q_0, q_1))} \right]^{m_1} \ln \left[ \int \mathcal{D}z_2 e^{-\beta V(\Xi(\delta, q_0, q_1))} \right]}{\int \mathcal{D}z_1 \left[ \int \mathcal{D}z_2 e^{-\beta V(\Xi(\delta, q_0, q_1))} \right]^{m_1}} \end{aligned} \quad (139)$$

and

$$\begin{aligned} \Sigma^*(m_1) &= \frac{1}{2} \ln[1 - m_1 q_0 + (m_1 - 1)q_1] - \frac{1}{2} \ln(1 - q_1) + \frac{\alpha}{p} \int \mathcal{D}z_0 \ln \left[ \int \mathcal{D}z_1 \left[ \int \mathcal{D}z_2 e^{-\beta V(\Xi(\delta, q_0, q_1))} \right]^{m_1} \right] \\ &+ m_1 \left\{ -\frac{1}{2} \left( -\frac{m_1 q_0}{[1 - m_1 q_0 + (m_1 - 1)q_1]^2} + \frac{1}{1 - m_1 q_0 + (m_1 - 1)q_1} \right) (q_1 - q_0) \right. \\ &\left. - \frac{\alpha}{p} \int \mathcal{D}z_0 \frac{\int \mathcal{D}z_1 \left[ \int \mathcal{D}z_2 e^{-\beta V(\Xi(\delta, q_0, q_1))} \right]^{m_1} \ln \left[ \int \mathcal{D}z_2 e^{-\beta V(\Xi(\delta, q_0, q_1))} \right]}{\int \mathcal{D}z_1 \left[ \int \mathcal{D}z_2 e^{-\beta V(\Xi(\delta, q_0, q_1))} \right]^{m_1}} \right\} \end{aligned} \quad (140)$$

### 3. $k = \infty$ case: continuous RSB

In the limit  $k \rightarrow \infty$ , the overlap matrix  $\hat{Q}$  is parametrized by function  $q(x)$  with  $n < x < 1$ . Then Eq. (116) becomes,

$$\ln \det \hat{Q}^{\infty\text{-RSB}} = \ln \left( 1 - \int_n^1 dx q(x) \right) - n \int_n^1 \frac{dx}{x^2} \ln \left( 1 - \int_x^1 dy q(y) - xq(x) \right) \quad (141)$$

From the above expression we find

$$\partial_n \ln \det \hat{Q}^{\infty\text{-RSB}} \Big|_{n=0} = \frac{q(0)}{G(0)} + \ln G(1) + \int_0^1 \frac{dx}{G(x)} \quad (142)$$

with

$$G(x) \equiv 1 - \int_x^1 dy q(y) - xq(x) \quad (143)$$

Then the free-energy Eq. (132) can be written as,

$$-\beta f_{\infty\text{-RSB}}[\hat{Q}] = \frac{1}{2} \left[ \frac{q(0)}{G(0)} + \frac{1}{2} \ln G(1) + \int_0^1 \frac{dx}{G(x)} \right] + \frac{\alpha}{p} \int \mathcal{D}z_0 (-f(m(0), \delta - \sqrt{q^p(0)}z_0)) \quad (144)$$

where the function  $f(x, h)$  obeys,

$$\dot{f}(x, h) = -\frac{1}{2} \dot{\lambda}(x) \left[ f''(x, h) - x (f'(x, h))^2 \right], \quad (145)$$

with

$$\lambda(x) \equiv q^p(x). \quad (146)$$

Here and in the following we denote a partial derivative with respect to the 1st argument by a dot, e. g.  $\partial_x f(x, h) = \dot{f}(x, h)$  and that with respect to the 2nd argument by a dash e. g.  $\partial_h f(x, h) = f'(x, h)$ . The partial differential equation Eq. (145) is the continuous limit of recursion formula Eq. (128). The boundary condition Eq. (129) becomes,

$$f(1, h) = -\ln \int \mathcal{D}z e^{-\beta V(h - \sqrt{1-q^p(1)}z)}. \quad (147)$$

### C. Variation of the interaction part of the free-energy

We will often meet the needs to consider variation of the free-energy. This happens for example when we wish to compute the pressure  $\Pi$  Eq. (68), the distribution of the gap  $g(r)$  Eq. (69) and also to obtain the saddle point equation for the order parameters  $q_i$ 's (sec. IV E). Here we consider a strategy to deal with the variation of the interaction part of the free-energy Eq. (120).

As we discussed in sec. IV B, the interaction part of the free-energy is constructed in a recursive way. This fact naturally motivates us to introduce for  $0 \leq i \leq j \leq k+1$ ,

$$P_{i,j}(y, h) \equiv \frac{\delta f(m_i, y)}{\delta f(m_j, h)}. \quad (148)$$

Using the chain rule we can write,

$$P_{i,j}(y, z) = \int dx P_{i,j-1}(y, x) P_{j-1,j}(x, z). \quad (149)$$

where

$$P_{j-1,j}(x, z) = \frac{\delta f(m_i, x)}{\delta f(m_{i+1}, z)} = e^{m_{j-1}(f(m_{j-1}, x) - f(m_j, z))} \frac{e^{-\frac{(x-z)^2}{2\Lambda_{j-1}}}}{\sqrt{2\pi\Lambda_{j-1}}} \quad (150)$$

as one can easily find from the recursion relation Eq. (128). Then we find the recursion relation,

$$P_{ij}(y, z) = e^{-m_{j-1}f(m_j, z)} \gamma_{\Lambda_{j-1}} \otimes_z \frac{P_{i, j-1}(y, z)}{e^{-m_{j-1}f(m_{j-1}, z)}} \quad (151)$$

with the 'boundary condition'

$$P_{ii}(y, h) = \delta(y - h). \quad (152)$$

Here  $\otimes_h$  stands for the convolution with respect to the variable  $h$ . A useful property to note is that the recursion relation Eq. (151) preserves the 'normalization',

$$\int dh P_{i,j}(y, h) = 1 \quad (153)$$

which can be easily proved using Eq. (128).

Now let us discuss how to analyze variations of the interaction part of the free-energy Eq. (120). The interaction part of the free-energy is given by Eq. (131) which reads as,  $-\partial_n \mathcal{F}_{\text{int}}[\hat{Q}^{k\text{-RSB}}] \Big|_{n=0} = -f(m_0 = n = 0, \delta) = -\int \mathcal{D}z_0 f(m_1, \delta - \sqrt{\Lambda_0} z_0)$ , except for the prefactor  $\alpha/p$ , it is convenient to introduce a short handed notation,

$$P(m_j, h) \equiv \frac{\delta f(m_0, \delta)}{\delta f(m_{j+1}, h)} = P_{0,j+1}(\delta, h) = \int dx P_{0,1}(\delta, x) P_{1,j+1}(x, h) = \int \mathcal{D}z_0 P_{1,j+1}(\delta - \sqrt{\Lambda_0} z_0, h) \quad (154)$$

where we used the chain rule, Eq. (150) and set  $m_0 = n \rightarrow 0$ . Clearly it follows the same recursion formula as Eq. (151),

$$P(m_j, h) = e^{-m_j f(m_{j+1}, h)} \gamma_{\Lambda_j} \otimes_h \frac{P(m_{j-1}, h)}{e^{-m_j f(m_j, h)}} \quad j = 1, 2, \dots, k+1 \quad (155)$$

with the 'boundary condition'

$$P(m_0, h) = \frac{1}{\sqrt{2\pi\Lambda_0}} e^{-\frac{(\delta-h)^2}{2\Lambda_0}} \quad (156)$$

which follows from Eq. (154) and Eq. (152). The functions  $P(m_i, h)$  is also normalized such that

$$\int dh P(m_i, h) = 1 \quad (157)$$

reflecting Eq. (153).

As an application, let us show that  $P(m_{k+1}, r)$  is nothing but the distribution function of the gap  $g(r)$  defined in Eq. (25). The distribution of the gap  $g(r)$  Eq. (69) can be obtained using Eq. (132), Eq. (130) which reads  $\beta V(r) = f(m_{k+2}, r)$ , Eq. (148), Eq. (154) and Eq. (155) as,

$$\begin{aligned} g(r) &= -\frac{p}{\alpha} \frac{\delta \beta f_{k\text{RSB}}[\hat{Q}^*]}{\delta(-\beta V(r))} = \int \mathcal{D}z_0 \frac{\delta f(m_1, \delta - \sqrt{\Lambda_0} z_0)}{\delta f(m_{k+2}, r)} \\ &= \int \mathcal{D}z_0 P_{1,k+2}(\delta - \sqrt{\Lambda_0} z_0, r) = P(m_{k+1}, r) = e^{-\beta V(r)} \gamma_{\Lambda_{k+1}} \otimes_r \frac{P(m_k, r)}{e^{-f(m_{k+1}=1, r)}} \end{aligned} \quad (158)$$

We used  $m_{k+1} = 1$  in the last equation. It can be seen that  $g(r)$  is properly normalized such that  $\int dr g(r) = 1$  reflecting Eq. (157). The previous result in the RS case ( $k = 0$ ) Eq. (83) can be recovered using Eq. (156), Eq. (128), Eq. (129) in Eq. (158).

In the  $k \rightarrow \infty$  limit, the function  $P(x, h)$  can be obtained by solving a differential equation,

$$\dot{P}(x, h) = \frac{1}{2} \dot{\lambda}(x) [P''(x, h) - 2x(P(x, h)\pi(x, h))'] \quad (159)$$

which is the continuous limit of Eq. (155). The boundary condition Eq. (156) becomes,

$$P(0, h) = \frac{1}{\sqrt{2\pi q^p(0)}} e^{-\frac{(\delta-h)^2}{2q^p(0)}}. \quad (160)$$

### D. Pressure

The pressure Eq. (68) for the generic  $k$ -RSB ansatz is obtained using and Eq. (132) as,

$$\Pi = -\frac{p}{\alpha} \frac{\partial \beta f_{k\text{-RSB}}[\hat{Q}^*]}{\partial \delta} = -\int \mathcal{D}z_0 \frac{\partial f(m_1, \delta - \sqrt{\Lambda_0} z_0)}{\partial \delta} = \int \mathcal{D}z_0 \pi(m_1, \delta - \sqrt{\Lambda_0} z_0). \quad (161)$$

where we introduced,

$$\pi(m, h) \equiv -f'(m, h) \quad (162)$$

Here and in the following the prime stands for taking a partial derivative with respect to the 2nd argument  $h$ . The function  $\pi(m, h)$  follows a recursion formula which can be obtained using Eq. (128) and Eq. (129) as,

$$\pi(m_i, h) = e^{m_i f(m_i, h)} \gamma_{\Lambda_i} \otimes \pi(m_{i+1}, h) e^{-m_i f(m_{i+1}, h)} \quad (163)$$

for  $i = 1, 2, \dots, k$  with the 'boundary condition',

$$\pi(m_{k+1}, h) = \frac{\int \mathcal{D}z_{k+1} (e^{-\beta V(h - \sqrt{\Lambda_{k+1}} z_{k+1})})'}{\int \mathcal{D}z_{k+1} e^{-\beta V(h - \sqrt{\Lambda_{k+1}} z_{k+1})}} \quad (164)$$

The previous result in the RS case ( $k = 0$ ) Eq. (82) can be recovered using Eq. (164) in Eq. (161).

It is also easy to see that the pressure Eq. (161) can be recovered through the virial equation for the pressure Eq. (26). In fact the pressure can be expressed as,

$$\Pi = \int dh P(m_{i-1}, h) \pi(m_i, h) \quad (165)$$

with *any*  $i = 1, 2, \dots, k+2$  (so that  $\pi = \int dh P(x, h) \pi(x, h)$  for any  $0 < x < 1$  in the  $k \rightarrow \infty$  limit). Using the recursion formulas Eq. (155) and Eq. (163) one can check that  $\int dh P(m_{i-1}, h) \pi(m_i, h) = \int dh' P(m_i, h') \pi(m_{i+1}, h')$  so that the r. h. s of the above equation does not depend on the level  $i$  of the hierarchy. The case of the virial equation for the pressure  $\Pi = \int dr g(r) (-\beta V'(r))$  Eq. (26) corresponds to the case  $i = k+2$  which can be seen by noting  $\pi(m_{k+2}, h) = -f'(m_{k+2}, h) = -\beta V'(h)$  (see Eq. (129)) and Eq. (158). On the other hand, the case  $i = 1$  corresponds to the expression Eq. (161) which can be seen using Eq. (156).

In the  $k \rightarrow \infty$  limit, the function  $\pi(x, h) \equiv -f'(x, h)$  can be obtained solving a differential equation,

$$\dot{\pi}(x, h) = -\frac{\dot{\lambda}(x)}{2} (\pi''(x, h) + 2x\pi(x, h)\pi'(x, h)) \quad (166)$$

which is the continuous limit of Eq. (163) and can be obtained from Eq. (145). The boundary condition for the latter is given by Eq. (164) which reads,

$$\pi(1, h) = \frac{\int \mathcal{D}z (e^{-\beta V(h - \sqrt{1-q^p(1)} z)})'}{\int \mathcal{D}z e^{-\beta V(h - \sqrt{1-q^p(1)} z)}}. \quad (167)$$

### E. Saddle point equations of the order parameter

Here we derive variational equations to determine  $q_i$  for  $i = 0, 1, 2, \dots, k$ . Since  $q_i$ 's are related to  $G_i$ 's through Eq. (119), we can consider,

$$\begin{aligned} 0 &= \frac{\partial(-\beta f_{k\text{-RSB}}[\hat{Q}])}{\partial G_i} \\ &= \frac{1}{2} \left[ -\frac{q_0}{G_0^2} \delta_{i,0} + \left( \frac{1}{m_{i+1}} - \frac{1}{m_i} \right) \left( \frac{1}{G_i} - \frac{1}{G_0} \right) (1 - \delta_{i,0}) \right] \\ &+ \left[ -\left( \frac{1}{m_{i+1}} - \frac{1}{m_i} \right) \sum_{j=0}^{i-1} p q_j^{p-1} \frac{\partial}{\partial \lambda_j} - \frac{1}{m_{i+1}} p q_i^{p-1} \frac{\partial}{\partial \lambda_i} \right] \frac{\alpha}{p} \int \mathcal{D}z_0 (-f(m_1, \delta - \sqrt{\Lambda_0} z_0)) \end{aligned} \quad (168)$$



In the last equation we used the fact that the interaction part depends on  $q_i$ 's through  $\lambda_i$ 's (see Eq. (122) and Eq. (121)). As we show in appendix B we find,

$$-\frac{\partial}{\partial \lambda_j} f(m_i, y) = \frac{1}{2}(m_j - m_{j+1}) \int dh P_{i,j+1}(y, h) \pi^2(m_{j+1}, h) \quad (169)$$

where  $P_{i,j}(y, h)$  is defined in Eq. (148) and  $\pi(m, h)$  is defined in Eq. (162)

Collecting the above results we obtain the variational equations as

$$\begin{aligned} \frac{q_0}{G_0^2} &= \kappa_0 \\ \frac{1}{G_i} - \frac{1}{G_0} &= \sum_{j=0}^{i-1} (m_j - m_{j+1}) \kappa_j + m_i \kappa_i \quad i = 1, 2, \dots, k \end{aligned} \quad (170)$$

where we introduced

$$\kappa_j \equiv \alpha q_j^{p-1} \int dh P(m_j, h) \pi^2(m_{j+1}, h). \quad (171)$$

where  $P(m_j, h)$  and  $\pi(m_i, h)$  can be obtained by solving the recursion formulas Eq. (155) and Eq. (163) respectively together with their boundary conditions.

We note that for  $p > 1$ ,  $q_0 = 0$  always solves the 1st equation of Eq. (170).

#### 1. $k = 0$ case: RS

Let us check if  $k = 0$  case recovers the result we obtained previously for the replica symmetric (RS) ansatz. In this case we just need the 1st equation of Eq. (170) which becomes,

$$\begin{aligned} \frac{q_0}{(1 - q_0)^2} &= \alpha q_0^{p-1} \int dh P(m_0, h) (\pi(m_1, h))^2 \\ &= \alpha q_0^{p-1} \int \mathcal{D}z_0 \left( \frac{\int \mathcal{D}z_1 (e^{-\beta V(x)})'}{\int \mathcal{D}z_1 e^{-\beta V(x)}} \right)^2 \Bigg|_{x=\delta - \sqrt{1-q_0^p} z_1 - \sqrt{q_0^p} z_0} \end{aligned} \quad (172)$$

where we used  $G_0 = 1 - (m_1 - m_0)q_0$  and that  $m_0 = 0$  and  $m_1 = 1$ . In the 2nd equation we used Eq. (164) and Eq. (156). The result agrees with Eq. (80) as it should.

#### 2. $k = 1$ case: 1RSB

For the  $k = 1$  case (1RSB) we have,

$$\begin{aligned} \frac{q_0}{G_0^2} &= \kappa_0 \\ \frac{1}{G_1} - \frac{1}{G_0} &= m_1(\kappa_1 - \kappa_0) \end{aligned} \quad (173)$$

with

$$\begin{aligned} \kappa_0 &= \alpha q_0^{p-1} \int dh P(m_0, h) \pi^2(m_1, h) \\ \kappa_1 &= \alpha q_1^{p-1} \int dh P(m_1, h) \pi^2(m_2, h) \end{aligned} \quad (174)$$

After solving the above equations for  $G_0$  and  $G_1$ , the order parameters  $q_0$  and  $q_1$  can be obtained as (See Eq. (119)),

$$\begin{aligned} q_0 &= 1 - G_1 + \frac{1}{m_1}(G_1 - G_0) \\ q_1 &= 1 - G_1 \end{aligned} \quad (175)$$

To evaluate  $\kappa_0$  and  $\kappa_1$  in Eq. (174) we need more information. Suppose that we are given some initial guess for the values of  $q_0$  and  $q_1$ . Then we can recursively obtain functions  $f(m_2, h)$  and  $f(m_1, h)$  (see Eq. (129)) and Eq. (128)) as,

$$\begin{aligned} e^{-m_2 f(m_2, h)} &= \int \mathcal{D}z_2 e^{-\beta V(h - \sqrt{1 - q_1^p} z_2)} \\ e^{-m_1 f(m_1, h)} &= \int \mathcal{D}z_1 e^{-m_1 f(m_2, h - \sqrt{q_1^p - q_0^p} z_1)} \end{aligned} \quad (176)$$

where  $m_2 = 1$ . Similarly we can recursively obtain functions  $\pi(m_1, h)$  and  $\pi(m_2, h)$  (See Eq. (163) and Eq. (164)) as,

$$\begin{aligned} \pi(m_2, h) &= \frac{\int \mathcal{D}z_2 (e^{-\beta V(h - \sqrt{1 - q_1^p} z_2)})'}{\int \mathcal{D}z_2 e^{-\beta V(h - \sqrt{1 - q_1^p} z_2)}} \\ \pi(m_1, h) &= e^{m_1 f(m_1, h)} \int \mathcal{D}z_1 \pi(m_2, h') e^{-m_1 f(m_2, h')} \Big|_{h' = h - \sqrt{q_1^p - q_0^p} z_1} \end{aligned} \quad (177)$$

Next we can recursively obtain functions  $P(m_0, h)$  and  $P(m_1, h)$  (see Eq. (155) and Eq. (156)) as,

$$\begin{aligned} P(m_0, h) &= \frac{1}{\sqrt{2\pi q_0^p}} e^{-\frac{(\delta - h)^2}{2q_0^p}} \\ P(m_1, h) &= e^{-m_1 f(m_2, h)} \int \mathcal{D}z_1 P(m_0, h') e^{m_1 f(m_1, h')} \Big|_{h' = h - \sqrt{q_1^p - q_0^p} z_1} \end{aligned} \quad (178)$$

With these we are now readily to evaluate  $\kappa_0$  and  $\kappa_1$  using Eq. (174).

To sum up, we can evaluate the 1RSB solution numerically as follows: (0) make some initial guess for the values of  $q_0$  and  $q_1$  (1) obtain  $f(m_2, h) \rightarrow f(m_1, h)$  using Eq. (176) (2) obtain  $\pi(m_2, h) \rightarrow \pi(m_1, h)$  using Eq. (177) (3) obtain  $P(m_0, h) \rightarrow P(m_1, h)$  using Eq. (178) (4) solve for  $G_0$  and  $G_1$  using Eq. (173) and Eq. (174) (5) compute  $q_0$  and  $q_1$  using Eq. (175) (6) return to (1). The procedure has to be repeated until the solution converges.

We note that the parameter  $m_1$  remains. In order to study the equilibrium state  $m_1$  is fixed by the condition of vanishing complexity  $\Sigma(m_1) = 0$ . (See sec. IV B 2)

### 3. $k > 1$ case

The saddle point equations for a generic finite  $k$ -RSB ansatz with some fixed values of  $0 < m_1 < m_2 < \dots < m_k < 1$  can be solved numerically generalizing the procedure explained above.

### 4. $k = \infty$ case: continuous RSB

In the limit  $k \rightarrow \infty$ , the variational equations Eq. (170) become,

$$\frac{q(0)}{G^2(0)} = \kappa(0) \quad (179)$$

$$\frac{1}{G(x)} - \frac{1}{G(0)} = - \int_0^x dy \kappa(y) + x \kappa(x) \quad (180)$$

with

$$\kappa(x) \equiv \alpha q^{p-1}(x) \int dh P(x, h) \pi^2(x, h) \quad (181)$$

From the above equations we can derive some exact identities which become useful later. Taking a derivative with respect to  $x$  on both sides of Eq. (180) and using Eq. (181), Eq. (143), Eq. (159), Eq. (166), we find after some integrations by parts,

$$1 = \alpha(p-1)q^{p-2}(x)G^2(x) \int dh P(x, h) \pi^2(x, h) + \alpha p q^{2(p-1)}(x)G^2(x) \int dh P(x, h) (\pi'(x, h))^2 \quad (182)$$

Then taking another derivative on both sides of the above equation we find after some integrations by parts,

$$\begin{aligned}
0 = & (p-1)q^{p-3}(x)[(p-2)G^2(x) - 2q(x)xG(x)] \int dh P(x, h) \pi^2(x, h) \\
& + 3p(p-1)q^{2p-3}(x)G^2(x) \int dh P(x, h) (\pi'(x, h))^2 \\
& + pq^{2(p-1)}(x)(-2xG(x)) \int dh P(x, h) (\pi'(x, h))^2 \\
& + p^2q^{3(p-1)}(x)G^2(x) \int dh P(x, h) [(\pi''(x, h))^2 - 2x(\pi'(x, h))^3].
\end{aligned} \tag{183}$$

## F. Stability of the kRSB solution

Stability of the  $k(\geq 1)$ -RSB ansatz must be examined by studying the eigenvalues of the Hessian matrix discussed in appendix A. As we note in sec A 2, there is a residual replica symmetry within each of the inner-core part of the replica groups. Here we do not study the complete spectrum of the eigenmodes of the Hessian matrix but focus on the replicon eigenvalue responsible for the replica symmetry breaking of the residual replica symmetry.

### 1. $k = 1$ case: 1RSB

For the  $k = 1$  case we find from Eq. (A39) using

$$\lambda_R = \frac{2}{(1-q_1)^2} - 2\frac{\alpha}{p} \int dh P(m_1, h) \left[ p(p-1)q^{p-2}(\pi(m_2, h))^2 + (pq^{p-1})^2(\pi'(m_2, h))^2 \right] \tag{184}$$

where  $m_2 = 1$ . Here we used  $\pi(x, h) \equiv -f'(x, h)$  defined in Eq. (162). The functions  $\pi(m_2, h)$  and  $P(m_1, h)$  are given by Eq. (177) and Eq. (178) respectively.

The vanishing on  $\lambda_R$  signals the Gardner's transition [29]: instability to further breaking of the replica symmetry.

### 2. $k = \infty$ case: continuous RSB

From Eq. (A39) we find for  $k = \infty$ , by which  $m_k \rightarrow 1$ ,

$$\lambda_R = \frac{2}{G(1)^2} - 2\frac{\alpha}{p} \int dh P(1, h) \left[ p(p-1)q^{p-2}(\pi(1, h))^2 + (pq^{p-1})^2(\pi'(1, h))^2 \right] \tag{185}$$

where we used  $\pi(x, h) \equiv -f'(x, h)$  defined in Eq. (162) and  $G(1) = 1 - q(1)$  which follows from Eq. (143). Now using the exact identity Eq. (182) which holds for the continuous RSB system, we find it vanishes exactly:  $\lambda_R = 0$ . Thus the fullRSB solution is marginally stable.

## V. GLASS PHASE OF HARDCORE MODEL

### A. Hardcore potential

For the hardcore potential Eq. (22) we find

$$f(m_{k+1}, h) = -\ln \Theta \left( \frac{h}{\sqrt{2\Lambda_{k+1}}} \right) \tag{186}$$

where  $\Theta(x)$  is defined in Eq. (89). Then

$$\pi(m_{k+1}, h) = \frac{1}{\sqrt{2\Lambda_{k+1}}} \frac{\Theta' \left( \frac{h}{\sqrt{2\Lambda_{k+1}}} \right)}{\Theta \left( \frac{h}{\sqrt{2\Lambda_{k+1}}} \right)} \tag{187}$$

The functions  $f(m_i, h)$  and  $\pi(m_i, h)$  are determined via recursion formulas Eq. (128) and Eq. (163) using the boundary values obtained above.

It is useful to study the asymptotic behavior of the functions  $f(m_i, h)$  and  $\pi(m_i, h)$  in the limit  $h \rightarrow -\infty$  both for numerical and analytical purposes. Using Eq. (89) and Eq. (91) and the recursion formula Eq. (128) one finds for  $i = 1, 2, \dots, k+1$ ,

$$f(m_i, h) = \begin{cases} 0 & h \rightarrow \infty \\ \frac{h^2}{2\tilde{\Lambda}_i} & h \rightarrow -\infty \end{cases} \quad \pi(m_i, h) = \begin{cases} 0 & h \rightarrow \infty \\ -\frac{h}{\tilde{\Lambda}_i} & h \rightarrow -\infty \end{cases}$$

where we introduced,

$$\tilde{\Lambda}_i \equiv \sum_{j=i}^{k+1} m_j \Lambda_j. \quad (188)$$

Note that  $\tilde{\Lambda}_{k+1} = m_{k+1} \Lambda_{k+1} = \Lambda_{k+1} = 1 - q_k^p$ . In the continuous limit  $k \rightarrow \infty$  this implies,

$$f(x, h) = \begin{cases} 0 & h \rightarrow \infty \\ \frac{h^2}{2\tilde{\Lambda}(x)} & h \rightarrow -\infty \end{cases} \quad \pi(x, h) = \begin{cases} 0 & h \rightarrow \infty \\ -\frac{h}{\tilde{\Lambda}(x)} & h \rightarrow -\infty \end{cases} \quad (189)$$

with

$$\tilde{\Lambda}(x) = 1 - \int_x^1 dy \lambda(y) - x \lambda(x). \quad (190)$$

with  $\lambda(x) = q^p(x)$  defined in Eq. (146).

The above observation suggests us to introduce a function  $j(m_i, h)$  defined as,

$$-f(m_i, h) \equiv -\frac{h^2}{2\tilde{\Lambda}_i} \theta(-h) + j(m_i, h). \quad (191)$$

From Eq. (128) we find that the function  $j(m_i, h)$  follows a recursion relation,

$$j(m_i, h) = \frac{1}{m_i} \ln \int dy K_{i,i+1}(h, y) e^{m_i j(m_{i+1}, y)} \quad (192)$$

with

$$K_{i,i+1}(y, h) \equiv \frac{1}{\sqrt{2\pi\tilde{\Lambda}_i}} \exp \left[ -\frac{(h-y)^2}{2\tilde{\Lambda}_i} - \frac{m_i}{2} \frac{y^2}{\tilde{\Lambda}_{i+1}} \theta(-y) + \frac{m_i}{2} \frac{h^2}{\tilde{\Lambda}_i} \theta(-h) \right] \quad (193)$$

and the boundary condition,

$$j(m_{k+1}, h) = \ln \Theta \left( \frac{h}{\sqrt{2\tilde{\Lambda}_{k+1}}} \right) + \frac{h^2}{2\tilde{\Lambda}_{k+1}} \theta(-h) \quad (194)$$

Correspondingly one finds that Eq. (163) becomes,

$$\pi(m_i, h) = \int dy \pi(m_{i+1}, y) K_{i,i+1}(h, y) e^{m_i(j(m_{i+1}, y) - j(m_i, y))} \quad (195)$$

## B. Jamming

Let us discuss properties of the system approaching the jamming. We expect the  $q(x)$  function of the full RSB solution has a continuous part for some range  $x_m < x < x_1$  and a plateau  $q(x) = q_1$  for  $x_1 < x < 1$ . In the hardcore model jamming means disappearance of the thermal fluctuation  $q_1 \rightarrow 1$ . For a convenience we define,

$$\Delta(x) \equiv 1 - q(x). \quad (196)$$

Then jamming means  $\Delta_1 = 1 - q_1 \rightarrow 0$ . We discuss below properties of the system encoded in the full RSB solution in the vicinity of the core  $x \rightarrow x_1$  which encodes physical properties of the system in the deepest part of the energy landscape.

1. *Scaling ansatz at the core  $x \rightarrow x_1$  in the jamming limit  $\Delta_1 \rightarrow 1$ .*

Following [11] and [19] we consider the following scaling ansatz at the core  $x \rightarrow x_1$  in the jamming limit  $\Delta_1 \rightarrow 0$ ,

$$\Delta(x)/\Delta_1 \simeq (x/x_1)^{-\kappa} \quad (197)$$

with an exponent  $\kappa$ .

From Eq. (159) and Eq. (166) we have,

$$\dot{P}(x, h) = \frac{\dot{\lambda}(x)}{2} [P''(x, h) - 2x(P(x, h)\pi(x, h))'] \quad (198)$$

$$\dot{\pi}(x, h) = -\frac{\dot{\lambda}(x)}{2} [\pi''(x, h) + 2x\pi(x, h)\pi'(x, h)] \quad (199)$$

Based on the asymptotic behavior of the function  $\pi(x, h)$  given in Eq. (189) we expect,

$$P(x, h) \simeq \frac{1}{\sqrt{2\pi\lambda(x)}} e^{-(\delta-h)^2/2\lambda(x)} \quad h \rightarrow +\infty \quad (200)$$

and

$$\dot{P}(x, h) = \frac{\dot{\lambda}(x)}{2} \left[ P''(x, h) + 2\frac{x}{\tilde{\Lambda}(x)} (P'(x, h)h + P(x, h)) \right] \quad h \rightarrow -\infty \quad (201)$$

For  $x \rightarrow x_1$  and  $\Delta_1 \rightarrow 0$  we can assume,

$$\dot{\lambda}(x) \simeq -p\dot{\Delta}(x) \quad \tilde{\Lambda}(x) \simeq pG(x) \quad G(x) \simeq \frac{\kappa}{\kappa-1} x\Delta(x) \quad (202)$$

which follow from Eq. (146), Eq. (190), Eq. (196) and Eq. (197). Then assuming  $P(x, h) \simeq A(x)e^{B(x)h-C(x)h^2/2}$  for  $x \rightarrow x_1$  one finds,  $A \propto \Delta^{-(1-1/\kappa)}$ ,  $B \propto \Delta^{-(1-1/\kappa)}$  and  $C \propto \Delta^{-2(1-1/\kappa)}$ . This implies the following scaling form for  $x \rightarrow x_1$ ,

$$P(x, h) \sim \Delta^{-\frac{\kappa-1}{\kappa}} P_0(h\Delta^{-\frac{\kappa-1}{\kappa}}) \quad h \rightarrow -\infty \quad (203)$$

with some scaling function  $P_0(x)$ .

To sum up we can expect the following three regimes [19] [11] for  $x \rightarrow x_1$ :

(0)  $h \rightarrow -\infty$ : Eq. (203) and Eq. (189) implies

$$P(x, h) \sim \Delta^{-c/\kappa} P_0(h\Delta^{-c/\kappa}) \quad \pi(x, h) \sim -\frac{h}{\tilde{\Lambda}(x)} \quad (204)$$

with

$$c = \kappa - 1 \quad (205)$$

(1)  $h \sim 0$

$$P(x, h) \sim \Delta^{-a/\kappa} P_1(h\Delta^{-b/\kappa}) \quad \pi(x, h) \sim \frac{\Delta^{b/\kappa}}{\tilde{\Lambda}(x)} \pi_1(h\Delta^{-b/\kappa}) \quad (206)$$

(2)  $h \rightarrow \infty$ : Eq. (200) ( $\lambda(x) \rightarrow 1$  for  $x \rightarrow x_1$  and  $q(1) \rightarrow 1$ ) and Eq. (189) implies

$$P(x, h) \sim P_2(h) \quad \pi(x, h) \sim 0 \quad (207)$$

In the above equations  $P_0(x), P_1(x), \pi_1(x)$  and  $P_2(x)$  are some smooth functions and  $a, b, c, \kappa$  are some exponents. In the following we assume that these exponents are positive.

Now we can make the following observations:

1. Matching between (0) and (1): assuming

$$P_0(u) \propto u^\theta \quad u \rightarrow 0 \quad (208)$$

$$P_1(u) \propto u^\theta \quad u \rightarrow -\infty \quad (209)$$

the following relation is needed,

$$\Delta^{-c/\kappa} (h \Delta^{-c/\kappa})^\theta \sim \Delta^{-a/\kappa} (h \Delta^{-b/\kappa})^\theta \quad (210)$$

which implies

$$\theta = \frac{c-a}{b-c}. \quad (211)$$

We also find

$$\pi_1(u) \sim -u \quad u \rightarrow -\infty \quad (212)$$

must hold.

2. Matching between (1) and (2): assuming

$$P_1(z) \propto z^{-\alpha} \quad z \rightarrow \infty \quad (213)$$

$$P_2(z) \propto z^{-\alpha} \quad z \rightarrow 0 \quad (214)$$

we find the following relation is needed to eliminate the dependence on  $\Delta$ ,

$$\alpha = \frac{a}{b} \quad (215)$$

3. Analysis on the intermediate regime  $h \sim 0$ : Plugging in Eq. (206) in Eq. (198) and using Eq. (202) we find, the contribution from the 1st term on the r.h.s. scales are  $(\Delta^{-b/\kappa})^2$  while those from the 2nd term on the r.h.s and the term on the l.h.s scales like  $\Delta^{-1}$ . Thus in order to have a non-trivial solution we need,

$$\frac{b}{\kappa} = \frac{1}{2}. \quad (216)$$

by which we can eliminate  $b$ . Now we are left with two exponents  $a$  and  $c = \kappa - 1$ . Furthermore plugging in Eq. (206) in Eq. (198) and Eq. (199) we find the following two ordinary differential equations,

$$\frac{a}{\kappa} P_1(z) + \frac{z}{2} P_1'(z) = \frac{p}{2} P_1''(z) - \frac{c}{\kappa} (P_1(z) \pi_1(z))' \quad (217)$$

$$\left(\frac{1}{2} - \frac{c}{\kappa}\right) \pi_1(z) - \frac{1}{2} z \pi_1'(z) = \frac{p}{2} \pi_1''(z) + \frac{c}{\kappa} \pi_1(z) \pi_1'(z) \quad (218)$$

which are subjected to the boundary condition

$$P_1(z) = \begin{cases} z^\theta & z \rightarrow -\infty \\ z^{-\alpha} & z \rightarrow \infty \end{cases} \quad \pi_1(z) = \begin{cases} -z & z \rightarrow -\infty \\ 0 & z \rightarrow \infty \end{cases} \quad (219)$$

One can check that the differential equations Eq. (218) with the boundary condition Eq. (219) is consistent with the scaling relations for  $\theta$  and  $\alpha$  given by Eq. (211) and Eq. (215).

Here we notice that the apparent dependence on  $p$  in Eq. (218) can be formally eliminated by the following replacement,

$$\frac{z}{\sqrt{p}} \rightarrow z \quad \frac{P_1(z)}{\sqrt{p}} \rightarrow P_1(z) \quad \frac{\pi_1(z)}{\sqrt{p}} \rightarrow \pi_1(z) \quad (220)$$

This means that if we find a solution for the  $p = 1$  case, the solutions for other values of  $p$  can be obtained using Eq. (220) in the reversed manner. Importantly such a solution satisfies the same desired asymptotic behaviors Eq. (219). This implies the universality does not change with  $p$ .



However as pointed out in [11] the above equations do not completely solve the problem. We are left with the exponent  $a$  undetermined while other quantities  $P_1(z)$ ,  $\pi_1(z)$  and the exponent  $c$  can be obtained in a form parametrized by  $a$ . (All other exponents are fixed given  $a$  and  $c$ .) The final task to fix the value of the exponent  $a$  can be done using the exact identity Eq. (183) which reads in the limit  $x \rightarrow x_1$  and  $q_1 = q(x_1) \rightarrow 1$

$$0 = (p-1) \int dh T_1(h) + \int dh T_2(h) \quad (221)$$

where we defined,

$$T_1(h) \equiv [(p-2)G^2(x_1) - 2x_1 G(x_1)]P(x_1, h)\pi^2(x_1, h) + 3pG^2(x_1)P(x_1, h)(\pi'(x_1, h))^2 \quad (222)$$

$$T_2(h) \equiv p(-2x_1 G(x_1))P(x_1, h)(\pi'(x_1, h))^2 + p^2 G^2(x_1)P(x_1, h) \left[ (\pi''(x_1, h))^2 - 2x_1 (\pi'(x_1, h))^3 \right]. \quad (223)$$

We notice that the contribution of  $\int dh T_1(h)$  vanishes for the  $p = 1$  case. Thus we must carefully examine whether  $\int dh T_1(h)$  remain relevant in the jamming limit  $\Delta_1 = \Delta(x_1) \rightarrow 0$  or not. We examine contributions of the integrals  $\int dh T_1(h)$  and  $\int dh T_2(h)$  from the regime (1)  $h \rightarrow -\infty$  and (2)  $h \sim 0$ . In the regime (3)  $h \rightarrow \infty$   $\pi(x, h) \sim 0$  Eq. (207) so we do not need to consider the regime (3). Using Eq. (204), Eq. (205), and Eq. (202) we find for the regime (0)  $h \rightarrow -\infty$ ,

$$\begin{aligned} \int_{\text{regime}(0)} dh T_1(h) &\sim -\frac{2}{p^2} x_1 \frac{c}{\kappa} \Delta_1^{-(1-c)/\kappa} \int_0^\infty dt P_0(-t) t^2 \\ \int_{\text{regime}(0)} dh T_2(h) &\sim 0 \end{aligned} \quad (224)$$

where we took leading terms for the jamming limit  $\Delta_1 \rightarrow 0$ . Similarly using Eq. (206), Eq. (216) and Eq. (202) we find for the regime (1)  $h \sim 0$ ,

$$\begin{aligned} \int_{\text{regime}(1)} dh T_1(h) &\sim \Delta^{1/2-a/\kappa} \int_{-\infty}^\infty dz P_1(z) \left[ \frac{3}{p} (\pi'_1(z))^2 - \frac{2}{p^2} x_1 \frac{c}{\kappa} \pi_1^2(z) \right] \\ \int_{\text{regime}(1)} dh T_2(h) &\sim \Delta^{-(1+a)/\kappa} \int_{-\infty}^\infty dz P_1(z) \left[ (\pi''(z))^2 - 2 \frac{c}{\kappa} \frac{1}{p} \left\{ (\pi'_1(z))^3 + (\pi'(z))^2 \right\} \right] \end{aligned} \quad (225)$$

Collecting the above results we find the relevant contribution in the jamming limit  $\Delta_1 \rightarrow 0$  is given by  $\int_{\text{regime}(1)} dh T_2(h)$ . It means that we must satisfy,

$$\int_{-\infty}^\infty dz P_1(z) \left[ (\pi''(z))^2 - 2 \frac{c}{\kappa} \frac{1}{p} \left\{ (\pi'_1(z))^3 + (\pi'(z))^2 \right\} \right] = 0 \quad (226)$$

Again we find the apparent  $p$  dependence can be formally eliminated by the replacement Eq. (220).

Based on the above analysis we can conclude that the critical exponents and the scaling functions  $P_1(z)$ , and  $\pi_1(z)$  does not dependent on  $p$ , i. e. super-universal.

## 2. Divergence of the pressure

The pressure can be expressed as Eq. (165) which reads,

$$\Pi = \int dh P(x, h) \pi(x, h) \quad (227)$$

where  $x$  can be chosen arbitrary. Using the scaling ansatz Eq. (204) and Eq. (202) at the core  $x \rightarrow x_1$  and jamming  $\Delta_1 \rightarrow 0$  we find contribution from largely negative region of  $h$  becomes

$$\int_{-\infty}^0 dh \left( -\frac{1}{p} \right) \frac{\kappa-1}{\kappa} \frac{h}{\Delta_1} \Delta^{-c/\kappa} P_0(h \Delta_1^{-c/\kappa}) \sim c_{\text{nt}} \Delta^{-1/\kappa} \quad c_{\text{nt}} = \frac{1}{p} \frac{\kappa-1}{\kappa} \int_0^\infty dt P_0(-t) t. \quad (228)$$

Similarly we can analyze contribution from the region  $h \sim 0$  using Eq. (206), and Eq. (202)

$$\int dh \frac{1}{p} \frac{h}{\Delta_1} \Delta_1^{-a/\kappa} P_1(h \Delta^{-b/\kappa}) \propto \Delta^{-a/\kappa}. \quad (229)$$

If  $a < 1$ , which is the case, the latter gives a only subdominant contribution. To sum up we find, the 'cage size'  $\Delta_1$  vanishing in the jamming limit  $\Pi \rightarrow \infty$  as,

$$\Delta_1 \propto \Pi^{-\kappa} \quad (230)$$

### 3. Distribution of gap

For the hardcore model the distribution of the gap  $g(r)$  within the  $k$ -RSB ansatz Eq. (158) reads,

$$g(r) = \theta(r) \int \mathcal{D}z \frac{P(m_k, r - \sqrt{1 - q_k^p} z)}{\Theta\left(\frac{r - \sqrt{1 - q_k^p} z}{\sqrt{2(1 - q_k^p)}}\right)} \quad (231)$$

1. For *fixed* finite  $r$ , sending  $q_k \rightarrow 1$ , we find,

$$g(r) = \theta(r) P(m_k, r) \quad (232)$$

where we used  $\lim_{X \rightarrow \infty} \Theta(X) = 1$ . This is a generalization of the RS ( $k = 0$ ) result Eq. (109).

In the  $k \rightarrow \infty$  limit, the scaling behavior of  $P(x, h)$  close to the core  $x \rightarrow x_1$  as described by Eq. (204) and Eq. (206) in the region vanishing in the jamming limit  $\Delta_1 \rightarrow 0$  implies development of a delta peak  $\delta(r)$ . On the other hand, we have the scaling behavior  $P(x, h) \sim h^{-\alpha}$  for fixed  $h \sim 0^+$  as given by Eq. (214) with  $\alpha = a/b$  Eq. (215). These observations implies,

$$g(r) \sim \delta(r) + c_{\text{nt}} \theta(r) r^{-\alpha}, \quad (233)$$

where  $c_{\text{nt}}$  is some numerical factor.

2. In the vanishing region around  $r = 0$  parametrized as  $r = (1 - q_k^p)\lambda$  we find, Assuming  $q_k \sim 1$  we find for  $r > 0$ ,

$$g(r) \sim \frac{1}{1 - q_k^p} \int_0^\infty dy P(m_k, -y) y e^{-\lambda y} \quad \lambda = \frac{r}{1 - q_k^p} \quad (234)$$

This is a generalization of the RS ( $k = 0$ ) result Eq. (109).

Now in the  $k \rightarrow \infty$  limit we have the scaling behavior  $P(x, h) \sim \Delta^{-c/\kappa} P_0(h \Delta^{-c/\kappa})$  for  $h < 0$  Eq. (204). Using this for  $x \rightarrow x_1$  we find,

$$g(r) \sim \frac{1}{p} \frac{1}{\Delta_1^{1/\kappa}} \int_0^\infty dt P_0(-t) t e^{-\frac{t}{p} \frac{r}{\Delta_1^{1/\kappa}}} \quad (235)$$

where we used  $c = \kappa - 1$  and  $1 - q_k^p \simeq p \Delta_1$  for  $\Delta_1 \rightarrow 0$ .

Using the above result we can evaluate the fraction of interactions or contacts which is closed. For any small but finite  $\epsilon$  we have,

$$\int_0^\epsilon dr g(r) = \int_0^\infty dt P_0(-t) \int_0^{\epsilon t / (p \Delta_1^{1/\kappa})} ds e^{-s} \xrightarrow{\Delta_1 \rightarrow 0} \int_0^\infty dt P_0(-t) \quad (236)$$

Thus in the jamming limit, the fraction of closed contact Eq. (27) can be expressed as,

$$f_{\text{closed}} = \lim_{\epsilon \rightarrow 0} \int_0^\epsilon dr g(r) = \int_0^\infty dt P_0(-t) \quad (237)$$

#### 4. Isostaticity

Let us consider whether isostaticity holds in the jamming limit. The condition of isostaticity Eq. (28) becomes in  $M \rightarrow \infty$  limit with  $\alpha = c/M$  fixed at jamming  $\Delta_1 \rightarrow 0$  becomes,

$$1 = \frac{\alpha}{p} \int_0^\infty dt P_0(-t) \quad (238)$$

where we used Eq. (237).

Actually using the exact identity Eq. (182) which holds for the continuous RSB together with the scaling behavior Eq. (204) in the  $h < 0$  region and the relation  $\tilde{\Lambda}(x) \simeq pG(x)$  given by the 2nd equation of Eq. (202) which hold close to the core  $x \rightarrow x_1$  at jamming  $\Delta_1 \rightarrow 0$  we find,

$$1 = \frac{\alpha(p-1)}{p^2} \int_{-\infty}^0 dh \Delta_1^{-c/\kappa} P_0(h \Delta_1^{-c/\kappa}) h^2 + \frac{\alpha}{p} \int_{-\infty}^0 dh \Delta_1^{-c/\kappa} P_0(h \Delta_1^{-c/\kappa}) \xrightarrow{\Delta_1 \rightarrow 0} \frac{\alpha}{p} \int_0^\infty dt P_0(-t) \quad (239)$$

Thus we see that the isostaticity holds at jamming. Note that the term which is proportional to  $p-1$  apparently violates the isostaticity but it scales as  $\Delta_1^{2c/\kappa}$  and becomes irrelevant in the jamming limit  $\Delta_1 \rightarrow 0$  as long as  $c/\kappa > 0$ .

## VI. ASSEMBLY OF ANISOTROPIC PARTICLES

### A. Model

Let us consider an assembly of *anisotropic* particles in  $d$ -dimensional space interacting with each other through a two-body potential,

$$H = \sum_{i < j} V(\mathbf{r}_{ij}, \mathbf{S}_i, \mathbf{S}_j) \quad (240)$$

where  $\mathbf{r}_i$  and  $\mathbf{S}_i$  ( $i = 1, 2, \dots, N$ ) are  $d$ -dimensional vectors representing the position and *shape* of the particles. The latter is, for instance, the director of Janus particles. We assume the system is rotationally and translationally invariant and the potential is parametrized as,

$$V(\mathbf{r}_{12}, \mathbf{S}_1, \mathbf{S}_2) = V(r_{12}, \hat{\mathbf{r}}_{12} \cdot \mathbf{S}_1, \hat{\mathbf{r}}_{12} \cdot \mathbf{S}_2, \mathbf{S}_1 \cdot \mathbf{S}_2) \quad (241)$$

where  $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ ,  $\hat{\mathbf{r}} = \mathbf{r}/r$  and  $r = |\mathbf{r}|$ . In the following we set up a mean-field theoretical framework to describe liquid and glass states of such a system in the  $d \rightarrow \infty$  limit.

We denote the number density of the system as  $\rho = N/V$  with  $V$  being the volume of the container. For the colloids including the Janus particles it is useful to introduce the volume fraction  $\varphi = \rho(\Omega_d/d)(D/2)^d$  where  $D$  is the diameter of the particle and  $\Omega_d$  is the surface area of unit sphere in  $d$ -dimensions. We also introduce the reduced volume fraction  $\hat{\varphi} = 2^d \varphi / d$  which is useful to consider  $d \rightarrow \infty$  limit [44].

Let us introduce a generalized density field,

$$\rho(\mathbf{r}, \mathbf{S}) \equiv \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{S} - \mathbf{S}_i). \quad (242)$$

which is normalized such that

$$\int d^d r d\mathbf{S} \rho(\mathbf{r}, \mathbf{S}) = N. \quad (243)$$

By recalling the fact that in the  $d \rightarrow \infty$  limit so that the free-energy of liquids can be expressed exactly by 1st virial expansion we find [9],

$$-\beta \mathcal{F}[\rho(\mathbf{r}, \mathbf{S})] = - \int_{\mathbf{r}, \mathbf{S}} \rho(\mathbf{r}, \mathbf{S}) (\ln \rho(\mathbf{r}, \mathbf{S}) - 1) + \frac{1}{2} \int_{\mathbf{r}_1, \mathbf{S}_1, \mathbf{r}_2, \mathbf{S}_2} \rho(\mathbf{r}_1, \mathbf{S}_1) \rho(\mathbf{r}_2, \mathbf{S}_2) f(\mathbf{r}_{12}, \mathbf{S}_1, \mathbf{S}_2) \quad (244)$$

where we introduced a shorthanded notation  $\int_{\mathbf{r}, \mathbf{S}} \equiv \int d^d r \int_{\mathbf{S}} d\mathbf{S}$  and the Mayer function,

$$f(\mathbf{r}_{12}, \mathbf{S}_1, \mathbf{S}_2) = e^{-\beta V(\mathbf{r}_{12}, \mathbf{S}_1, \mathbf{S}_2)} \quad (245)$$

Here the potential  $V(\mathbf{r}_{12}, \mathbf{S}_1, \mathbf{S}_2)$  is rotationally and translationally invariant as in Eq. (241)

## B. Replicated liquid of anisotropic particles

Next we consider the replicated liquid of the anisotropic particles made of replicas  $a = 1, 2, \dots, m$  obeying the Hamiltonian,

$$H_m = \sum_{a=1}^m \sum_{i < j} V(\mathbf{r}_{ij}^a, \mathbf{S}_i^a, \mathbf{S}_j^a). \quad (246)$$

We denote the position of the particles as  $\bar{\mathbf{r}}_i = (\mathbf{r}_i^1, \mathbf{r}_i^2, \dots, \mathbf{r}_i^m)$  as Eq. (35) By introducing the replicated density field,

$$\rho(\bar{\mathbf{r}}, \bar{\mathbf{S}}) = \sum_{i=1}^N \prod_{a=1}^m \delta(\mathbf{r}^a - \mathbf{r}_i^a) \delta(\mathbf{S}^a - \mathbf{S}_i^a) \quad (247)$$

we can write the free-energy of the replicated system as,

$$-\beta F_m[\rho(\bar{\mathbf{r}}, \bar{\mathbf{S}})] = - \int_{\bar{\mathbf{r}}, \bar{\mathbf{S}}} \rho(\bar{\mathbf{r}}, \bar{\mathbf{S}}) (\ln \rho(\bar{\mathbf{r}}, \bar{\mathbf{S}}) - 1) + \frac{1}{2} \int_{\bar{\mathbf{r}}_1, \bar{\mathbf{S}}_1, \bar{\mathbf{r}}_2, \bar{\mathbf{S}}_2} \rho(\bar{\mathbf{r}}_1, \bar{\mathbf{S}}_1) \rho(\bar{\mathbf{r}}_2, \bar{\mathbf{S}}_2) f_m(\bar{\mathbf{r}}_{12}, \bar{\mathbf{S}}_1, \bar{\mathbf{S}}_2) \quad (248)$$

where we introduced the replicated Mayer function,

$$f_m(\bar{\mathbf{r}}_{12}, \bar{\mathbf{S}}_1, \bar{\mathbf{S}}_2) = \prod_{a=1}^m e^{-\beta V(\mathbf{r}_{12}^a, \mathbf{S}_1^a, \mathbf{S}_2^a)} - 1 \quad (249)$$

Here the potential  $V(\bar{\mathbf{r}}_{12}, \bar{\mathbf{S}}_1, \bar{\mathbf{S}}_2)$  is rotationally and translationally invariant as in Eq. (241).

As in the case of simple spheres [9] we decompose the spatial coordinate of particles as,

$$\mathbf{r}_i^a = \mathbf{R}_i + \mathbf{u}_i^a \quad (250)$$

where

$$\mathbf{R}_i = \frac{1}{m} \sum_{a=1}^m \mathbf{r}_i^a \quad (251)$$

is the center of mass position of the molecule made of replicas  $a = 1, 2, \dots, m$  and  $\mathbf{u}_i^a$  represents fluctuation within the molecule. The natural glass order parameters which are invariant under global translations and rotations of all replicas is,

$$q_{ab} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \langle \mathbf{u}_i^a \cdot \mathbf{u}_i^b \rangle. \quad (252)$$

$$Q_{ab} = \lim_{N \rightarrow \infty} \frac{1}{Nd} \sum_{i=1}^N \langle \mathbf{S}_i^a \cdot \mathbf{S}_i^b \rangle. \quad (253)$$

$$P_{ab} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \langle \mathbf{u}_i^a \cdot \mathbf{S}_i^b \rangle. \quad (254)$$

Note that the 2nd equation which defines  $Q_{ab}$  is the same as Eq. (40) (using  $M = d$ ). For convenience we define a combined matrix as shown in Fig. 10 and call it as  $\hat{Q}_{\text{tot}}$  of size  $2m \times 2m$ .  $\hat{Q}_{\text{tot}}$  is parametrized completely by a smaller matrix  $\hat{Q}_{\text{tot}}^{mm}$  which is defined by subtracting the  $m$ -th row and column (shaded region in Fig. 10).

We expect the  $\rho(\bar{\mathbf{r}}, \bar{\mathbf{S}})$  of the glass states which keep the translational and rotational invariance of the liquid can be parametrized solely by the glass order parameters as,

$$\rho(\bar{\mathbf{r}}, \bar{\mathbf{S}}) = \rho(\hat{Q}_{\text{tot}}) \quad (255)$$

Now we wish to find the exact expression of the free-energy  $-\beta F/N$  of the molecular liquid of the anisotropic particles in terms of the glass order parameters  $(q_{\text{tot}})_{ab}$  combining the result of the simple spheres [11] and our result of the vectorial spin system reported in sec. II B 2 and sec. II B 3.

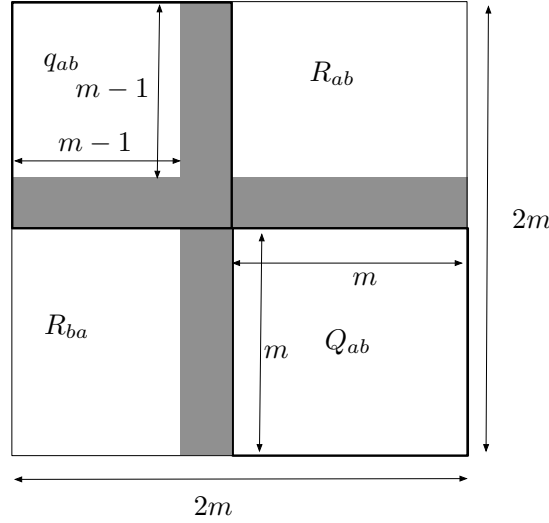


FIG. 10. Parametrization of the extended Parisi's matrix  $\hat{Q}_{\text{tot}}$

The 1st step is to change the integration variables in Eq. (248) from  $\bar{\mathbf{r}}, \bar{\mathbf{S}}$  to  $\hat{Q}_{\text{tot}}$ . Because of the decomposition Eq. (250) let us change integration variables as,  $\int_{\bar{\mathbf{r}}} \dots = \int d^d R \int \mathcal{D}\bar{\mathbf{u}} \dots$  with  $\mathcal{D}\bar{\mathbf{u}} \equiv d^d \bar{\mathbf{u}} m^d \delta(\sum_{a=1}^m \mathbf{u}^a)$ . Subsequently let us change the integration variables from  $(\bar{\mathbf{u}}, \bar{\mathbf{S}})$  to  $\hat{Q}_{\text{tot}}$  defined in Eq. (254) introducing the Jacobian

$$j(\hat{Q}_{\text{tot}}) \equiv \int \mathcal{D}\bar{\mathbf{u}} d\bar{\mathbf{S}} \prod_{a \leq b}^{1,m} \delta \left( q_{ab} - \sum_{\mu=1}^d (u^a)^\mu (u^b)^\mu \right) \prod_{a \leq b}^{1,m} \delta \left( Q_{ab} - \frac{1}{d} \sum_{\mu=1}^d (S^a)^\mu (S^b)^\mu \right) \\ \times \prod_{a=1}^m \prod_{b=1}^m \delta \left( P_{ab} - \sum_{\mu=1}^d (u^a)^\mu (S^b)^\mu \right) \quad (256)$$

Now the steps down to Eq. (59) is much the same here so we do not repeat those. The free-energy as a functional of the order parameters consists of two parts : the entropic term and the interaction term, which are the first two terms and last term on the right hand side of Eq. (59). The entropic term is readily obtained as in [9–11] (see equation (1) of [11]),

$$-\frac{\beta F_{\text{ent}}}{N} = 1 - \ln \rho + d \ln m + \frac{(2m-1)d}{2} \ln \left( \frac{2\pi e}{d} \right) + \frac{d}{2} \ln \det \hat{Q}_{\text{tot}}^{m,m}. \quad (257)$$

Next let us analyze the interaction part of the free-energy, i. e. the 2nd term on the r.h.s of Eq. (248) which becomes,

$$-\frac{\beta F_{\text{int}}}{N} = \frac{1}{2\rho} \int d\hat{Q}_{\text{tot},1} d\hat{Q}_{\text{tot},2} j(\hat{Q}_{\text{tot},1}) j(\hat{Q}_{\text{tot},2}) \rho(\hat{Q}_{\text{tot},1}) \rho(\hat{Q}_{\text{tot},2}) \bar{f}(\hat{Q}_{\text{tot},1}, \hat{Q}_{\text{tot},2}) \quad (258)$$

Assuming the rotationally and translationally invariant potential Eq. (241) we can write,

$$\bar{f}(\hat{Q}_{\text{tot},1}, \hat{Q}_{\text{tot},2}) \equiv \int d^d R \left\langle f_m \left( |\bar{\mathbf{r}}_1 - \bar{\mathbf{r}}_2|^2, \hat{\mathbf{R}} \cdot \mathbf{S}_1^a, \hat{\mathbf{R}} \cdot \mathbf{S}_2^a, \mathbf{S}_1^a \cdot \mathbf{S}_2^a \right) \right\rangle_{\hat{Q}_{\text{tot}}} \quad (259)$$

$$= \int_0^\infty dR R^{d-1} \int d\Omega_d \left\langle f_m(R^2 + 2R\hat{\mathbf{R}} \cdot (\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2) + (\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2)^2, \hat{\mathbf{R}} \cdot \mathbf{S}_1^a, \hat{\mathbf{R}} \cdot \mathbf{S}_2^a, \mathbf{S}_1^a \cdot \mathbf{S}_2^a) \right\rangle_{\hat{Q}_{\text{tot}}} \\ = \int_0^\infty dR R^{d-1} \int \frac{d\bar{\lambda}}{2\pi} \frac{d\bar{\gamma}}{2\pi} \frac{d\bar{\gamma}'}{2\pi} \frac{d\bar{\kappa}}{2\pi} \tilde{f}_m(\bar{\lambda}, \bar{\gamma}, \bar{\gamma}', \bar{\kappa}) \left\langle e^{i \sum_{a=1}^m \kappa_a \mathbf{S}_1^a \cdot \mathbf{S}_2^a} e^{i \sum_{a=1}^m \lambda_a (\mathbf{u}_1^a - \mathbf{u}_2^a)^2} \right. \\ \left. \int d\Omega_d e^{i \hat{\mathbf{R}} \cdot \sum_{a=1}^m (2R\lambda_a (\mathbf{u}_1^a - \mathbf{u}_2^a) + \gamma_a \mathbf{S}_1^a + \gamma'_a \mathbf{S}_2^a)} \right\rangle_{\hat{Q}_{\text{tot}}} \quad (260)$$

Here  $\int d^d R$  is an integration over the separation of the center of mass between the molecules 1 and 2 and  $\hat{R} = \mathbf{R}/R$  is a unit vector whose direction represented by the solid angle  $\Omega$  is integrated over by  $\int d\Omega \dots$ . Here we introduced

$$\begin{aligned} \langle \dots \rangle_{\hat{Q}_{\text{tot}}} &\equiv \int \prod_{l=1,2} \left\{ d\bar{\mathbf{u}}_l d\bar{\mathbf{S}}_l \frac{1}{J(\hat{Q}_{\text{tot}}, l)} \prod_{a \leq b}^{1,m} \delta \left( (q_l)_{ab} - \frac{1}{d} \sum_{\mu=1}^d ((u_l)^a)^\mu ((u_l)^b)^\mu \right) \prod_{a \leq b}^{1,m} \delta \left( (Q_l)_{ab} - \frac{1}{d} \sum_{\mu=1}^d ((S_l)^a)^\mu ((S_l)^b)^\mu \right) \right. \\ &\times \left. \prod_{a=1}^m \prod_{b=1}^m \delta \left( (P_l)_{ab} = \sum_{\mu=1}^d ((u_l)^a)^\mu ((S_l)^b)^\mu \right) \right\} \dots \end{aligned} \quad (261)$$

We also introduced a Fourier transform,

$$f_m(R^2 + \bar{y}, \bar{x}, \bar{x}', \bar{h}) = \int \frac{d\bar{\lambda}}{2\pi} \frac{d\bar{\gamma}}{2\pi} \frac{d\bar{\gamma}'}{2\pi} \frac{d\bar{\kappa}}{2\pi} e^{i \sum_{a=1}^m (\lambda_a y_a + \gamma_a x_a + \gamma'_a x'_a + \kappa_a h_a)} \bar{f}_m(\bar{\lambda}, \bar{\gamma}, \bar{\gamma}', \bar{\kappa}). \quad (262)$$

The integration over the solid angle can be evaluated as follows. For arbitrary  $d$ -dimensional vectors  $\mathbf{A}_a$  we have,

$$\int d\Omega_d e^{i \hat{R} \cdot \sum_{a=1}^m \mathbf{A}_a} = \Omega_d \left\{ 1 - \frac{1}{2d} \sum_{\mu=1}^d \sum_{a,b=1}^m A_a^\mu A_b^\mu + O(1/d) \right\} \quad (263)$$

where  $\Omega_d$  is the solid angle in the  $d$ -dimensional space. This can be seen by noting  $\langle \hat{R}^\mu \rangle_\Omega = 0$ ,  $\langle \hat{R}^\mu \hat{R}^\nu \rangle_\Omega = (1/d) \delta_{\mu\nu}, \dots$  where we defined the average over the solid angle  $\langle \dots \rangle \equiv (1/\Omega_d) \int d\Omega_d \dots$ . As in the case of the  $M$ -component spin system in  $M \rightarrow \infty$  limit, we assume that different components  $\hat{R}^\mu$  become independent from each other in  $d \rightarrow \infty$  limit.

Using  $A_a = 2R\lambda_a(\mathbf{u}_1^a - \mathbf{u}_2^a) + \gamma_a \mathbf{S}_1^a + \gamma'_a \mathbf{S}_2^a$  in the above formula we find,

$$\begin{aligned} &\left\langle e^{i \sum_{a=1}^m \kappa_a \mathbf{S}_1^a \cdot \mathbf{S}_2^a} e^{i \sum_{a=1}^m \lambda_a (\mathbf{u}_1^a - \mathbf{u}_2^a)^2} \int d\Omega_d e^{i \hat{R} \cdot \sum_{a=1}^m [2R\lambda_a(\mathbf{u}_1^a - \mathbf{u}_2^a) + \gamma_a \mathbf{S}_1^a + \gamma'_a \mathbf{S}_2^a]} \right\rangle_{\hat{Q}_{\text{tot}}} \\ &= \exp \left( i \sum_{a=1}^m \lambda_a ((q_1)_{aa} + (q_2)_{aa}) - \frac{1}{2d} \sum_{a,b=1}^m (2R\lambda_a)(2R\lambda_b) ((q_1)_{ab} + (q_2)_{ab}) \right. \\ &\quad \left. - \frac{1}{d} \sum_{ab} (2R)\lambda_a (\gamma_b (P_1)_{ab} - \gamma'_b (P_2)_{ab}) - \frac{1}{2} \sum_{ab} \gamma_a \gamma_b (Q_1)_{ab} - \frac{1}{2} \sum_{a,b=1}^m \gamma'_a \gamma'_b (Q_2)_{ab} - \frac{1}{2} \sum_{a,b=1}^m \kappa_a \kappa_b (Q_1)_{ab} (Q_2)_{ab} \right) \end{aligned} \quad (264)$$

and we obtain,

$$\begin{aligned} \bar{f}(\hat{Q}_{\text{tot},1}, \hat{Q}_{\text{tot},2}) &= \Omega_d \int_0^\infty dR R^{d-1} \int d\bar{y} d\bar{x} d\bar{x}' d\bar{h} \left\{ \exp \left[ - \sum_{a=1}^m ((q_1)_{aa} + (q_2)_{aa}) \frac{\partial}{\partial y_a} + \frac{1}{2d} \sum_{a,b=1}^m \frac{\partial^2}{\partial y_a \partial y_b} (2R)^2 ((q_1)_{ab} + (q_2)_{ab}) \right] \right. \\ &\quad + \frac{2R}{d} \sum_{ab} \frac{\partial^2}{\partial y_a \partial x_b} (P_1)_{ab} - \frac{2R}{d} \sum_{ab} \frac{\partial^2}{\partial y_a \partial x'_b} (P_2)_{ab} + \frac{1}{2} \sum_{a,b=1}^m \frac{\partial^2}{\partial x_a \partial x_b} (Q_1)_{ab} + \frac{1}{2} \sum_{a,b=1}^m \frac{\partial^2}{\partial x'_a \partial x'_b} (Q_2)_{ab} \\ &\quad \left. + \frac{1}{2} \sum_{a,b=1}^m (Q_1)_{ab} (Q_2)_{ab} \frac{\partial^2}{\partial h_a \partial h_b} \right] \prod_{a=1}^m \{ \delta(y_a) \delta(x_a) \delta(x'_a) \delta(h_a) \} \} f_m(R^2 + \bar{y}, \bar{x}, \bar{x}', \bar{h}) \\ &= \Omega_d \int_0^\infty dR R^{d-1} \exp \left[ \sum_{a=1}^m ((q_1)_{aa} + (q_2)_{aa}) \frac{\partial}{\partial y_a} + \frac{1}{2d} \sum_{a,b=1}^m \frac{\partial^2}{\partial y_a \partial y_b} (2R)^2 ((q_1)_{ab} + (q_2)_{ab}) \right. \\ &\quad + \frac{2R}{d} \sum_{ab} \frac{\partial^2}{\partial y_a \partial x_b} (P_1)_{ab} - \frac{2R}{d} \sum_{ab} \frac{\partial^2}{\partial y_a \partial x'_b} (P_2)_{ab} + \frac{1}{2} \sum_{a,b=1}^m \frac{\partial^2}{\partial x_a \partial x_b} (Q_1)_{ab} + \frac{1}{2} \sum_{a,b=1}^m \frac{\partial^2}{\partial x'_a \partial x'_b} (Q_2)_{ab} \\ &\quad \left. + \frac{1}{2} \sum_{a,b=1}^m (Q_1)_{ab} (Q_2)_{ab} \frac{\partial^2}{\partial h_a \partial h_b} \right] f_m(R^2 + \bar{y}, \bar{x}, \bar{x}', \bar{h}) \Bigg|_{\bar{y}=\bar{x}=\bar{x}'=\bar{h}=0} \end{aligned} \quad (265)$$

$$(266)$$

Now we have to evaluate the integration  $\int dR R^{d-1} \dots$ . In order to take  $d \rightarrow \infty$  limit, it is useful to change the integration variable from  $R$  to a scaled variable  $\xi$  [11],

$$R \equiv D \left(1 + \frac{\xi}{d}\right) \quad (267)$$

with which we can write

$$\int_0^\infty dR R^{d-1} \xrightarrow{d \rightarrow \infty} \frac{D^d}{d} \int_{-\infty}^\infty d\xi e^\xi \quad (268)$$

then we obtain

$$\begin{aligned} \bar{f}(\hat{Q}_{\text{tot},1}, \hat{Q}_{\text{tot},2}) \Big|_{d \rightarrow \infty} &= \frac{D^d}{d} \Omega_d \int_0^\infty d\xi e^\xi \exp \left[ \frac{1}{2} \sum_a ((\alpha_1)_{aa} + (\alpha_2)_{aa}) \frac{\partial^2}{\partial \xi_a \partial \xi_b} + \frac{1}{2} \sum_{ab} ((\alpha_1)_{ab} + (\alpha_2)_{ab}) \frac{\partial^2}{\partial \xi_a \partial \xi_b} \right. \\ &+ \sum_{ab} \left( (\beta_1)_{ab} \frac{\partial^2}{\partial \xi_a \partial x_b} - (\beta_2)_{ab} \frac{\partial^2}{\partial \xi_a \partial x'_b} \right) \\ &+ \frac{1}{2} \sum_{ab} (Q_1)_{ab} \frac{\partial^2}{\partial x_a \partial x_b} + \frac{1}{2} \sum_{ab} (Q_2)_{ab} \frac{\partial^2}{\partial x_a \partial x_b} + \frac{1}{2} \sum_{ab} (Q_1)_{ab} (Q_2)_{ab} \frac{\partial^2}{\partial h_a \partial h_b} \left. \right] f_m \left( D^2 \left( 1 + \frac{\bar{\xi}}{d} \right)^2, \bar{x}, \bar{x}', \bar{h} \right) \Big|_{\bar{\xi}=\xi, \bar{y}=\bar{x}=\bar{x}'=\bar{h}=0} \\ &= \frac{D^d}{d} \Omega_d \int_0^\infty d\xi e^\xi \prod_{a,b=1}^m \exp \left[ -\frac{1}{4} ((\Delta_1)_{ab} + (\Delta_2)_{ab}) \frac{\partial^2}{\partial \xi_a \partial \xi_b} \right. \\ &+ \sum_{ab} \left( (\beta_1)_{ab} \frac{\partial^2}{\partial \xi_a \partial x_b} - (\beta_2)_{ab} \frac{\partial^2}{\partial \xi_a \partial x'_b} \right) \\ &+ \frac{1}{2} (Q_1)_{ab} \frac{\partial^2}{\partial x_a \partial x_b} + \frac{1}{2} (Q_2)_{ab} \frac{\partial^2}{\partial x_a \partial x_b} + \frac{1}{2} (Q_1)_{ab} (Q_2)_{ab} \frac{\partial^2}{\partial h_a \partial h_b} \left. \right] f_m \left( D^2 \left( 1 + \frac{\bar{\xi}}{d} \right)^2, \bar{x}, \bar{x}', \bar{h} \right) \Big|_{\bar{\xi}=\xi, \bar{y}=\bar{x}=\bar{x}'=\bar{h}=0} \end{aligned} \quad (269)$$

where we used  $R \rightarrow D$  in  $d \rightarrow \infty$  for finite  $\xi$  and introduced scaled order parameters,

$$\alpha_{ab} \equiv \frac{d}{D^2} q_{ab} \quad \beta_{ab} = \frac{1}{D} P_{ab} \quad (270)$$

In the last equation we introduced

$$\Delta_{ab} \equiv \alpha_{aa} + \alpha_{bb} - 2\alpha_{ab} \quad (271)$$

Let us comment on the derivation of the last equation of Eq. (269). There we assumed that the diagonal elements of the  $\hat{\alpha}$  matrix is a constant, say  $\alpha_{aa} = \alpha_d$  [11], which is independent of replicas  $a$ . This holds for the usual Parisi's replica symmetry breaking ansatz [39] which allows us to rewrite the  $\xi$  dependent part of the integrand as,

$$\begin{aligned} \int_{-\infty}^\infty d\xi e^\xi e^{\frac{1}{2} \sum_a \alpha_{aa} \frac{\partial^2}{\partial \xi_a \partial \xi_b} + \frac{1}{2} \sum_{ab} \alpha_{ab} \frac{\partial^2}{\partial \xi_a \partial \xi_b}} A(\bar{\xi}) \Big|_{\bar{\xi}=0} &= \int_{-\infty}^\infty d\xi e^\xi e^{\alpha_d (\sum_a \frac{\partial}{\partial \xi_a} + 1) \sum_b \frac{\partial}{\partial \xi_b}} e^{-\frac{1}{2} \sum_{ab} \Delta_{ab} \frac{\partial^2}{\partial \xi_a \partial \xi_b}} A(\bar{\xi}) \Big|_{\bar{\xi}=0} \\ &= \int_{-\infty}^\infty d\xi e^\xi e^{-\frac{1}{2} \sum_{ab} \Delta_{ab} \frac{\partial^2}{\partial \xi_a \partial \xi_b}} A(\bar{\xi}) \Big|_{\bar{\xi}=0} \end{aligned} \quad (272)$$

where  $A(\bar{\xi})$  represents the operand of the differential operators  $\partial_{\xi_a}$ . In the last step we repeated integrations by parts.

Assuming that  $\hat{Q}_{\text{tot},1}^* = \hat{Q}_{\text{tot},2}^*$  at the saddle point, which is defined by normalization conditions  $\rho = \int d\hat{Q}_{\text{tot}} \rho(\hat{Q}_{\text{tot}})$  we find the interaction part of the free-energy as,

$$-\frac{\beta F_{\text{int}}}{N} = \frac{d}{2} \hat{\varphi} \int_{-\infty}^\infty d\xi e^\xi e^{\frac{1}{2} \sum_{a,b=1}^m \mathcal{D}_{ab}} \left[ \prod_{a=1}^m e^{-\beta V(D(1+\xi_a/d), x_a, x'_a, y_a)} \Big|_{\substack{\{\xi_a=\xi\} \\ \{x_a, x'_a, h_a=0\}}} - 1 \right] \quad (273)$$

where we introduced

$$\mathcal{D}_{ab} = -\Delta_{ab} \frac{\partial^2}{\partial \xi_a \partial \xi_b} + \beta_{ab} \left( \frac{\partial}{\partial \xi_a} + \frac{\partial}{\partial \xi_b} \right) \left( \frac{\partial}{\partial x_b} - \frac{\partial}{\partial x'_b} \right) + Q_{ab} \left( \frac{\partial^2}{\partial x_a \partial x_b} + \frac{\partial^2}{\partial x'_a \partial x'_b} \right) + Q_{ab}^2 \frac{\partial^2}{\partial h_a \partial h_b} \quad (274)$$

and

$$\hat{\varphi} \equiv \rho 2^d \varphi / d \quad \varphi = \rho \frac{\Omega_d}{d} (D/2)^d \quad (275)$$

Including the entropic part of the free-energy we finally obtain,

$$-\beta \frac{F[\hat{Q}_{\text{tot}}]}{N} = 1 - \ln \rho + d \ln m + \frac{(2m-1)d}{2} \ln \left( \frac{2\pi e}{d} \right) + \frac{d}{2} \ln \det \hat{Q}_{\text{tot}}^{m,m} - \frac{d}{2} \hat{\varphi} \mathcal{F}[\hat{Q}_{\text{tot}}] \quad (276)$$

with

$$-\mathcal{F}[\hat{Q}_{\text{tot}}] \equiv \int_{-\infty}^{\infty} d\xi e^{\xi} e^{\frac{1}{2} \sum_{a,b=1}^m \mathcal{D}_{ab}} \left[ \prod_{a=1}^m e^{-\beta V(D(1+\frac{\xi_a}{d}), x_a, x'_a, h_a)} \Big|_{\substack{\{\xi_a=\xi\} \\ \{x_a, x'_a, h_a=0\}}} - 1 \right] \quad (277)$$

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### Appendix A: Eigenvalues of the stability matrix

Here we analyze the Hessian matrix  $M_{a \neq b, c \neq d}$  of the free-energy around the saddle points. It is a matrix of size  $n(n-1) \times n(n-1)$  defined as,

$$M_{a \neq b, c \neq d} \equiv -\frac{\partial^2 s[\hat{Q}]}{\partial Q_{a < b} \partial Q_{c < d}} \quad (\text{A1})$$

where  $s_n[\hat{Q}]$  is the free-entropy defined in Eq. (66) which reads,

$$s_n[\hat{Q}] \equiv \frac{1}{2} \ln \det \hat{Q} - \frac{\alpha}{p} \mathcal{F}_{\text{int}}[\hat{Q}] \quad (\text{A2})$$

$$-\mathcal{F}_{\text{int}}[\hat{Q}] \equiv \exp \left( \frac{1}{2} \sum_{a,b=1}^n (Q_{ab})^p \frac{\partial^2}{\partial h_a \partial h_b} \right) \prod_{a=1}^n e^{-\beta V(\delta + h_a)} \Big|_{\{h_a=0\}}. \quad (\text{A3})$$

The Hessian matrix can be naturally written as sum of the contribution from the entropic part and interaction part of the free-energy,

$$M_{a \neq b, c \neq d} = M_{a \neq b, c \neq d}^{\text{ent.}} + M_{a \neq b, c \neq d}^{\text{int.}} \quad (\text{A4})$$

with

$$M_{a \neq b, c \neq d}^{\text{ent.}} = -\frac{\partial^2}{\partial Q_{a < b} \partial Q_{c < d}} \frac{1}{2} \ln \det \hat{Q} = Q_{ac}^{-1} Q_{bd}^{-1} + Q_{ad}^{-1} Q_{bc}^{-1} \quad (\text{A5})$$

$$\begin{aligned} M_{a \neq b, c \neq d}^{\text{int.}} &= \frac{\alpha}{p} \frac{\partial^2}{\partial Q_{a < b} \partial Q_{c < d}} \mathcal{F}_{\text{int}}[\hat{Q}] \\ &= \frac{\alpha}{p} \left[ p(p-1) Q_{a < b}^{p-2} (\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \frac{\partial^2}{\partial h_a \partial h_b} \right. \\ &\quad \left. + p^2 Q_{a < b}^{p-1} Q_{c < d}^{p-1} \frac{\partial^4}{\partial h_a \partial h_b \partial h_c \partial h_d} \right] \mathcal{F}_{\text{int}}[\hat{Q}, \{h_a\}] \Big|_{\{h_a=0\}} \end{aligned} \quad (\text{A6})$$

with

$$-\mathcal{F}_{\text{int}}[\hat{Q}, \{h_a\}] \equiv \exp \left( \frac{1}{2} \sum_{e,f=1}^n Q_{ef}^p \frac{\partial^2}{\partial h_e \partial h_f} \right) \prod_{a=1}^n e^{-\beta V(\delta + h_a)} \quad (\text{A7})$$

#### 1. RS ansatz

Here we analyze the eigenvalues of the Hessian matrix for the case of the replica symmetric (RS) solution characterized by the order parameter matrix of the form Eq. (71), which reads as,

$$\hat{Q}^{\text{RS}} = (1-q)\delta_{ab} + q \quad (\text{A8})$$

The replica symmetry implies the following matrix structure,

$$M_{a \neq b, c \neq d} = M_1 \frac{\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}}{2} + M_2 \frac{\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd}}{4} + M_3 \quad (\text{A9})$$

from which the eigenvalues of the Hessian matrix are obtained as [10, 27],

$$\lambda_R = M_1 \quad (\text{A10})$$

$$\lambda_L = n(n-1)M_3 + (n-1)M_2 + M_1 \xrightarrow{n \rightarrow 0} M_1 - M_2 \quad (\text{A11})$$

$$\lambda_A = \frac{1}{2}(n-2)M_2 + M_1 \xrightarrow{n \rightarrow 0} M_1 - M_2 \quad (\text{A12})$$

The factors  $M_i$ 's can be decomposed into the entropic and interaction parts,  $M_i = M_i^{\text{ent}} + M_i^{\text{int}}$  like Eq. (A4).

First let us examine the entropic part. The replica symmetric matrix Eq. (A8) can be easily inverted to find,

$$(\hat{Q}^{\text{RS}})^{-1}_{ab} = \hat{q}\delta_{ab} + \tilde{q} \quad (\text{A13})$$

with

$$\hat{q} = \frac{1}{1-q} \quad (\text{A14})$$

$$\tilde{q} = -\frac{q}{1+(n-2)q-(n-1)q^2} \xrightarrow{n \rightarrow 0} -\frac{q}{(1-q)^2} \quad (\text{A15})$$

Using this in Eq. (A5), we obtain the entropic contributions as,

$$M_1^{\text{ent.}} = \lim_{n \rightarrow 0} 2(\hat{q})^2 = \frac{2}{(1-q)^2} \quad (\text{A16})$$

$$M_2^{\text{ent.}} = \lim_{n \rightarrow 0} 4\hat{q}\tilde{q} = -4\frac{q}{(1-q)^3} \quad (\text{A17})$$

$$M_3^{\text{ent.}} = \lim_{n \rightarrow 0} 2(\tilde{q})^2 = 2\frac{q^2}{(1-q)^4} \quad (\text{A18})$$

Next let us examine the interaction part Eq. (A6). Within the RS ansatz we find,

$$\begin{aligned} \lim_{n \rightarrow 0} M_{a \neq b, c \neq d}^{\text{int.}} = \lim_{n \rightarrow 0} \exp \left( \frac{q^p}{2} \sum_{e,f=1}^n \frac{\partial^2}{\partial h_e \partial h_f} \right) \frac{\alpha}{p} \left[ p(p-1)q^{p-2}(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \frac{\partial^2}{\partial h_a \partial h_b} \right. \\ \left. + p^2 q^{2(p-1)} \frac{\partial^4}{\partial h_a \partial h_b \partial h_c \partial h_d} \right] \prod_{a=1}^n g(\delta + h_a) \Big|_{\{h_a=0\}} \end{aligned} \quad (\text{A19})$$

where we used

$$\begin{aligned} -\mathcal{F}_{\text{int}}[\hat{Q}^{\text{RS}}, \{h_a\}] &= \exp \left( \frac{1}{2} \sum_{e,f=1}^n ((1-q^p)\delta_{ab} + q^p) \frac{\partial^2}{\partial h_e \partial h_f} \right) \prod_{a=1}^n e^{-\beta V(\delta + h_a)} \\ &= \exp \left( \frac{q^p}{2} \sum_{e,f=1}^n \frac{\partial^2}{\partial h_e \partial h_f} \right) \prod_{a=1}^n g(\delta + h_a) \end{aligned} \quad (\text{A20})$$

In the last equation we introduced a shorthand notation of the quantity defined in Eq. (123),

$$g(h) \equiv g(m_{k+1}, h) = \gamma_{1-q_k^p} \otimes e^{-\beta V(h)} \quad (\text{A21})$$

For a convenience let us also introduce a related shorthand notation (See Eq. (127))

$$f(h) \equiv f(m_{k+1}, h) = -\frac{1}{m_{k+1}} \log g(m_{k+1}, h) = -\log g(h) \quad (\text{A22})$$

where  $m_{k+1} = 1$ .

By taking derivatives we find,

$$\begin{aligned} \lim_{n \rightarrow 0} \exp \left( \frac{q^p}{2} \sum_{e,f=1}^n \frac{\partial^2}{\partial h_e \partial h_f} \right) \frac{\partial^2}{\partial h_a \partial h_b} \prod_{a=1}^n g(\delta + h_a) \Big|_{h_a=0} \\ = \exp \left( \frac{q^p}{2} \frac{\partial^2}{\partial h^2} \right) \left( \frac{g'(h)}{g(h)} \right)^2 \Big|_{h=\delta} = \gamma_{q^p} \otimes \left( \frac{g'(\delta)}{g(\delta)} \right)^2 \end{aligned} \quad (\text{A23})$$

in the last equation we used Eq. (75). Similarly we obtain,

$$\begin{aligned}
& \lim_{n \rightarrow 0} \exp \left( \frac{q^p}{2} \sum_{e,f=1}^n \frac{\partial^2}{\partial h_e \partial h_f} \right) \frac{\partial^4}{\partial h_a \partial h_b \partial h_c \partial h_d} \prod_{a=1}^n g(\delta + h_a) \Big|_{\{h_a=0\}} \\
&= \gamma_{q^p} \otimes \left\{ (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \left( \frac{g''(\delta)}{g(\delta)} \right)^2 + [\delta_{ac} + \delta_{bc} + \delta_{ad} + \delta_{bd} - 2(\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd})] \left[ \frac{g''(\delta)}{g(\delta)} \left( \frac{g'(\delta)}{g(\delta)} \right)^2 \right] \right. \\
&\quad \left. + [1 - (\delta_{ac} + \delta_{bc} + \delta_{ad} + \delta_{bd}) + (\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd})] \left( \frac{g'(\delta)}{g(\delta)} \right)^4 \right\} \tag{A24}
\end{aligned}$$

From the above result we find the contributions by the interaction part as,

$$\begin{aligned}
-M_1^{\text{int.}} &= \frac{2\alpha}{p} \left[ p(p-1)q^{p-2}\gamma_{q^p} \otimes \left( \frac{g'(\delta)}{g(\delta)} \right)^2 + (pq^{p-1})^2\gamma_{q^p} \otimes \left\{ \left( \frac{g''(\delta)}{g(\delta)} \right)^2 - 2\frac{g''(\delta)}{g(\delta)} \left( \frac{g'(\delta)}{g(\delta)} \right)^2 + \left( \frac{g'(\delta)}{g(\delta)} \right)^4 \right\} \right] \\
&= \frac{2\alpha}{p} \left[ p(p-1)q^{p-2}\gamma_{q^p} \otimes (f'(\delta))^2 + (pq^{p-1})^2\gamma_{q^p} \otimes (f''(\delta))^2 \right] \\
-M_2^{\text{int.}} &= \frac{4\alpha}{p} (pq^{p-1})^2\gamma_{q^p} \otimes \left\{ \frac{g''(\delta)}{g(\delta)} \left( \frac{g'(\delta)}{g(\delta)} \right)^2 - \left( \frac{g'(\delta)}{g(\delta)} \right)^4 \right\} = \frac{4\alpha}{p} (pq^{p-1})^2\gamma_{q^p} \otimes (-f''(\delta)(f'(\delta))^2) \\
-M_3^{\text{int.}} &= \frac{\alpha}{p} (pq^{p-1})^2\gamma_{q^p} \otimes \left( \frac{g'(\delta)}{g(\delta)} \right)^4 = \frac{\alpha}{p} (pq^{p-1})^2\gamma_{q^p} \otimes (f'(\delta))^4 \tag{A25}
\end{aligned}$$

## 2. $k$ -RSB ansatz

Next let us analyze the case of  $k$ -step replica symmetry breaking solution with the ansatz Eq. (111). Within the  $k$ -RSB ansatz,  $n$  replicas are divided into  $n/m_1$  groups of size  $m_1$  and each of the latter is divided into  $m_1/m_2$  groups of size  $m_2$ , and so on. Finally we find  $n/m_k$  groups of size  $m_k$ . Within each of the groups of size  $m_k$ , the replica symmetry remains. As we did in the 1-RSB case, here we only analyze stability of the replica symmetry within such a most inner-core group. Thus we just consider the Hessian matrix  $M_{a \neq b, c \neq d}$  Eq. (A1) assuming that all indexes  $a, b, c, d$  are in the same most-inner core replica group of size  $m_k$ , which we denote as  $\mathcal{C}$  in the following.

### a. Contributions from the interaction term

Let us first examine the contributions from the interaction term. Within the  $k$ -RSB ansatz, the interaction part of the Hessian matrix  $M_{a \neq b, c \neq d}^{\text{int.}}$  Eq. (A4) for  $a, b, c, d$  in the same most-inner core replica group  $\mathcal{C}$  becomes, using Eq. (120) and Eq. (126),

$$\begin{aligned}
& - \lim_{n \rightarrow 0} M_{a \neq b, c \neq d}^{\text{int., } \mathcal{C}} = \lim_{n \rightarrow 0} \prod_{l=0}^k \exp \left( \frac{\Lambda_l}{2} \sum_{e,f=1}^n I_{ef}^{m_l} \frac{\partial^2}{\partial h_e \partial h_f} \right) \left( \prod_{a \notin \mathcal{C}} g(\delta + h_a) \right) \frac{\alpha}{p} \left[ p(p-1)q_k^{p-2}(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \frac{\partial^2}{\partial h_a \partial h_b} \right. \\
&\quad \left. + p^2 q_k^{2(p-1)} \frac{\partial^4}{\partial h_a \partial h_b \partial h_c \partial h_d} \right] \prod_{a \in \mathcal{C}} g(\delta + h_a) \Big|_{\{h_a=0\}} \\
&= \lim_{m_0 \rightarrow 0} \gamma_{\Lambda_0} \otimes \left\{ g^{m_0/m_1-1}(m_1, \delta) \gamma_{\Lambda_1} \otimes \left\{ g^{m_1/m_2-1}(m_2, \delta) \gamma_{\Lambda_2} \otimes \{ \dots \right. \right. \\
&\quad \left. \left. \dots g^{m_{k-1}/m_k-1}(m_k, h) \gamma_{\Lambda_k} \otimes \left\{ g^{m_k}(m_{k+1}, h) \left[ S_1(h) \frac{\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}}{2} + S_2(h) \frac{\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd}}{4} + S_3(h) \right] \right\} \right\} \Big|_{h=\delta} \\
&= \int dh P(m_k, h) \left[ S_1(h) \frac{\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}}{2} + S_2(h) \frac{\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd}}{4} + S_3(h) \right] \tag{A26}
\end{aligned}$$

In the last equation we used Eq. (??) derived in appendix ?? and Eq. (154). In the last equation we introduced,

$$S_1(h) = \frac{2\alpha}{p} \left[ p(p-1)q^{p-2}(f'(h))^2 + (pq^{p-1})^2(f''(h))^2 \right] \quad (\text{A27})$$

$$S_2(h) = \frac{4\alpha}{p} (pq^{p-1})^2 (-f''(h)(f'(h))^2) \quad (\text{A28})$$

$$S_3(h) = \frac{\alpha}{p} (pq^{p-1})^2 (f'(h))^4 \quad (\text{A29})$$

Thus we find the contributions from the interaction term as,

$$-M_1^{\text{int.}} = \int dh P(m_k, h) S_1(h) \quad (\text{A30})$$

$$-M_2^{\text{int.}} = \int dh P(m_k, h) S_2(h) \quad (\text{A31})$$

$$-M_3^{\text{int.}} = \int dh P(m_k, h) S_3(h) \quad (\text{A32})$$

The above formula reduces to the RS one Eq. (A19) for  $k = 0$  case as it should.

### b. Contributions from the entropic term

Next let us examine the entropic contribution. To this end it is useful to note first that the entropic contribution to the  $k$ -RSB free-energy can also be expressed in a recursive manner exploiting the hierarchical structure of the order parameter,

$$\frac{1}{2} \ln \det \hat{Q} = -\ln I(\hat{Q}) \quad (\text{A33})$$

$$I(\hat{Q}) \equiv \int \prod_{a=1}^n d\phi_a e^{-\frac{1}{2} \sum_{a,b=1}^n \phi_a Q_{ab} \phi_b + \sum_a h_a \phi_a} \Bigg|_{h_a=0} = \prod_{l=0}^k e^{-\frac{\Lambda_l}{2} \sum_{ab} I_{ab}^{m_l} \frac{\partial^2}{\partial h_a \partial h_b}} \prod_{a=1}^n g_e(m_{k+1}, h_a) \quad (\text{A34})$$

where (see Eq. (111))

$$\Lambda_0 = q_0 \quad \Lambda_i = q_i - q_{i+1} \quad (\text{A35})$$

and

$$g_e(m_{k+1}, h) \equiv \int \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}(1-q_k)\phi^2 + h\phi} = \frac{e^{\frac{h^2}{2(1-q_k)}}}{\sqrt{1-q_k}} \quad (\text{A36})$$

Comparing the above expressions with Eq. (120) we find the entropic term is expressed very similarly as the interaction term. We just need to put  $p = 1$  in Eq. (122) (see also Eq. (121)) and replace  $g(m_{k+1}, h)$  by  $g_e(m_{k+1}, h)$  defined above. Again we can define a family of functions  $g_e(m_l, h)$  for  $l = 0, 1, \dots, k$  through Eq. (124) with the boundary condition Eq. (A36). Then we can write  $\ln I(\hat{Q}) = g_e(m_0, 0)$ . In addition we can introduce  $f_e(m_i, h) \equiv -(1/m_i) \ln g_e(m_i, h)$  (Eq. (127)) and  $P_{i,j}^e(y, h) \equiv \delta f_e(m_i, y) / \delta f_e(m_j, h)$  (Eq. (148)) and  $P^e(m_j, h) \equiv P_{0,j+1}^e(0, h)$  (Eq. (154)).

We have to note however that sign in front of  $\Lambda_l$  in Eq. (A34) is *negative*. Thus we have to understand the operator  $\otimes$  which appears in equations like Eq. (124) not in the Gaussian convolution form Eq. (77) but in the original differential form Eq. (75).

Using the above results we can write the entropic part of the submatrix of the Hessian matrix associated with a most inner core group  $\mathcal{C}$  as,

$$\begin{aligned} \lim_{n \rightarrow 0} M_{a \neq b, c \neq d}^{\text{ent., } \mathcal{C}} &= - \lim_{n \rightarrow 0} \frac{\partial^2}{\partial Q_{a \neq b} \partial Q_{c \neq d}} \frac{1}{2} \ln \det \hat{Q} \\ &= \lim_{n \rightarrow 0} Q_{a \neq b}^{-1} Q_{c \neq d}^{-1} + \lim_{n \rightarrow 0} \frac{\partial^2}{\partial Q_{a \neq b} \partial Q_{c \neq d}} I(\hat{Q}) \end{aligned} \quad (\text{A37})$$

Note that the 1st term on the r.h.s of the last equation contributes only to  $M_3^{\text{ent.}}$ . For the replicon mode we need  $M_1^{\text{ent.}}$  which is obtained as,

$$M_1^{\text{ent.}} = \frac{2}{(1 - q_k)^2}. \quad (\text{A38})$$

This can be obtained using Eq. (A30) and Eq. (A27) with the following modifications:  $p \rightarrow 1$ ,  $f''(\delta) \rightarrow f''_e(m_{k+1}, 0) = -1/(1 - q_k)$  which can be obtained from Eq. (A36), and  $-\alpha/p \rightarrow 1$ .

### c. Replicon eigenvalue

Summing up the above results we find the replicon eigenvalue which is responsible for the RSB instability of a most-inner core replica group in the  $k$ -RSB ansatz as,

$$\lambda_R = \frac{2}{(1 - q_k)^2} - 2\frac{\alpha}{p} \int dh P(m_k, h) \left[ p(p-1)q^{p-2}(f'(m_{k+1}, h))^2 + (pq^{p-1})^2(f''(m_{k+1}, h))^2 \right] \quad (\text{A39})$$

## Appendix B: Derivation of Eq. (169)

Here we show the derivation of Eq. (169). Let us begin with the case  $1 \leq i = j \leq k$ . Using the recursion formula Eq. (128) we find,

$$\partial_{\lambda_i} f(m_i, y) = e^{m_i f(m_i, y)} \int \mathcal{D}z_i e^{-m_i f(m_{i+1}, \Xi_i)} \partial_{\lambda_i} f(m_{i+1}, \Xi_i) \quad (\text{B1})$$

where  $\Xi_i = y - \sqrt{\lambda_i - \lambda_{i-1}} z_i$  and with  $\Xi_{i+1} = \Xi_i - \sqrt{\lambda_{i+1} - \lambda_i} z_{i+1}$  we find,

$$\partial_{\lambda_i} f(m_{i+1}, y) = e^{m_{i+1} f(m_{i+1}, y)} \int \mathcal{D}z_{i+1} e^{-m_{i+1} f(m_{i+2}, \Xi_{i+1})} \partial_{\lambda_i} f(m_{i+2}, \Xi_{i+1}) \quad (\text{B2})$$

Then by noting that  $\Xi_{i+1} = y - \sqrt{\lambda_i - \lambda_{i-1}} z_i - \sqrt{\lambda_{i+1} - \lambda_i} z_{i+1}$  we find,

$$\partial_{\lambda_i} f(m_{i+2}, \Xi_{i+1}) = f'(m_{i+2}, \Xi_{i+1}) \left( \frac{1}{2} \frac{z_{i+1}}{\sqrt{\lambda_{i+1} - \lambda_i}} - \frac{1}{2} \frac{z_i}{\sqrt{\lambda_i - \lambda_{i-1}}} \right) \quad (\text{B3})$$

where the dash represents the partial derivative with respect to the 2nd argument  $\partial_h f(x, h) = f'(x, h)$ .

Collecting the above results, we find for  $i = 0, 2, \dots, k$ ,

$$\begin{aligned} \partial_{\lambda_i} f(m_i, y) &= \frac{1}{2} (m_{i+1} - m_i) e^{m_i f(m_i, y)} \int \mathcal{D}z_i e^{-m_i f(m_{i+1}, \Xi_i)} (f'(m_{i+1}, \Xi_i))^2 \\ &= \frac{1}{2} (m_{i+1} - m_i) \int dh P_{i,i+1}(y, h) (f'(m_{i+1}, h))^2. \end{aligned} \quad (\text{B4})$$

To derive the 1st equation we performed integrations by parts. In 2nd equation we used the identity Eq. (150). One can naturally generalize the above analysis and find for  $1 \leq i \leq j \leq k$ ,

$$\partial_{\lambda_j} f(m_i, y) = \int dh \frac{\delta f(m_i, y)}{\delta f(m_j, h)} \partial_{\lambda_j} f(m_j, h) = \frac{1}{2} (m_j - m_{j+1}) \int dh P_{j,j+1}(y, h) (f'(m_{j+1}, h))^2. \quad (\text{B5})$$

which is the desired result Eq. (169). In the last equation we used Eq. (148), Eq. (B4) and the identity Eq. (149).

### Appendix C: Expansion of $P_{0,j}(h, y)$

Using the recursion formula Eq. (125) (see also the expansion displayed in Eq. (126)) we find,

$$\begin{aligned}
\frac{\delta g(m_0, h)}{\delta g(m_j, y)} &= \frac{m_0}{m_1} \gamma_{\Lambda_0} \otimes \left\{ g^{m_0/m_1-1}(m_1, h) \frac{\delta g(m_1, h)}{\delta g(m_j, y)} \right\} \Big|_{h=\delta} \\
&= \frac{m_0}{m_1} \frac{m_1}{m_2} \gamma_{\Lambda_0} \otimes \left\{ g^{m_0/m_1-1}(m_1, h) \gamma_{\Lambda_1} \otimes \left\{ g^{m_1/m_2-1}(m_2, h) \frac{\delta g(m_2, h)}{\delta g(m_j, y)} \right\} \right\} \Big|_{h=\delta} \\
&= \dots \\
&= \frac{m_0}{m_1} \frac{m_1}{m_2} \dots \frac{m_{j-1}}{m_j} \gamma_{\Lambda_0} \otimes \left\{ g^{m_0/m_1-1}(m_1, h) \gamma_{\Lambda_1} \otimes \left\{ g^{m_1/m_2-1}(m_2, h) \gamma_{\Lambda_2} \otimes \dots \right. \right. \\
&\quad \left. \left. \dots \gamma_{\Lambda_{j-1}} \otimes \left\{ g^{m_{j-1}/m_j-1}(m_j, h) \delta(h-y) \right\} \right\} \right\}
\end{aligned} \tag{C1}$$

On the other hand using Eq. (148) and Eq. (127) we find,

$$\frac{\delta g(m_0, h)}{\delta g(m_j, y)} = \frac{m_0}{m_j} \frac{g(m_0, \delta)}{g(m_j, y)} P_{0,j}(\delta, y). \tag{C2}$$

Combining the above results we find,

$$\begin{aligned}
P_{0,j}(h, y) &= \gamma_{\Lambda_0} \otimes \left\{ g^{m_0/m_1-1}(m_1, h) \gamma_{\Lambda_1} \otimes \left\{ g^{m_1/m_2-1}(m_2, h) \gamma_{\Lambda_2} \otimes \dots \right. \right. \\
&\quad \left. \left. \dots \gamma_{\Lambda_{j-1}} \otimes \left\{ g^{m_{j-1}/m_j-1}(m_j, h) \delta(h-y) \right\} \right\} \right\}.
\end{aligned} \tag{C3}$$

In the last equation we have took the limit  $m_0 = n \rightarrow 0$  so that  $\lim_{m_0 \rightarrow 0} g(m_0, h) \rightarrow 1$ .

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